Introduction to Quantum Groups

"A quantum group is a Hopf algebra obtained by deformation of the universal enveloping algebra of a Lie algebra."

§0. Informal Motivation

†Geka (2014) / Pek & Yang (2006)

Motto Quantum groups exist as a machinery to produce solutions to the Yang-Baxter equation:

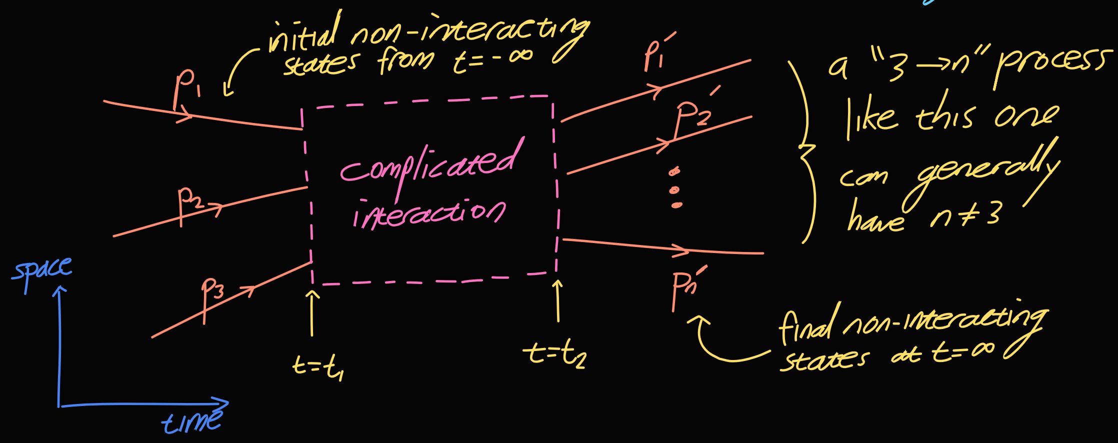
Let R be a 2×2 matrix. The "Yang-Baxter" eq specifies

$$(R\otimes id)(id\otimes R)(R\otimes id) = (id\otimes R)(R\otimes id)(id\otimes R)$$

This eq appears all over physics; the quintessential example is calculation of factorizable S-matrices for many-particle scattering

Example (S-matrix) In QFT, we often care about scattering interaxns

eg



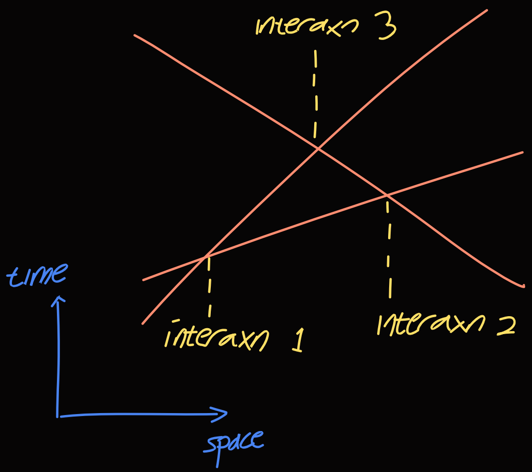
The quantity of physical interest is the differential scattering cross sxn, $\frac{d\sigma}{d\Omega}$, which is entirely given by matrix elements of the so-called S-matrix

"transfer matrix"

$$\langle p_1p_2p_3|S|p_1'\cdots p_n'\rangle$$

$$= \langle p_1p_2p_3|\, I + i(2\pi)^4\delta^4(\Sigma p_i - \Sigma p_i')\,\mathcal{M}\,|p_1'\cdots p_n'\rangle$$

In the simplest case, the particle # is unchanged, + S is "factorizable" – the n-particle interaxn decomposes into a sequence of independent 2-particle interaxns governed by 2×2 matrix R; schematically:



$R\otimes id$

$id\otimes R$

$R\otimes id$

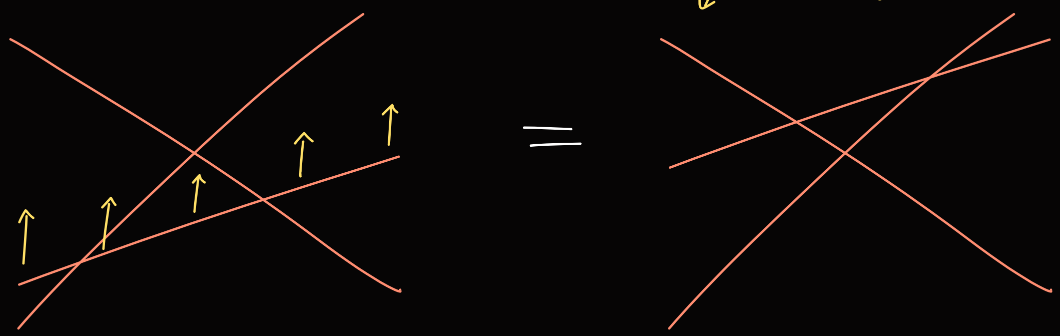
$S = (R\otimes id)(id\otimes R)(R\otimes id)$

Vertical comp is $\circ$ (juxtapos)

Horizontal comp is $\otimes$

Such scenarious exhibit symmetry under reordering of pairwise interaxns. ie.

resembles Reidemeister move!



$$(R\otimes id)(id\otimes R)(R\otimes id) = (id\otimes R)(R\otimes id)(id\otimes R)$$

§1. Universal Enveloping Algebra

† Hilgert & Neeb (2012) §7.1

Fix Lie alg $\mathfrak{g} \equiv (T_1 G, [\cdot,\cdot]_{\mathfrak{g}})$

Idea Embed $\mathfrak{g}$ in a unital assoc alg $\mathcal{U}(\mathfrak{g})$ st

① $\mathcal{U}(\mathfrak{g})$ is generated by elmts of $\mathfrak{g}$

② $[\cdot,\cdot]_{\mathfrak{g}}$ becomes commutator in $\mathcal{U}(\mathfrak{g})$

Remark This is how we think of matrix Lie alg's

Def (univ env alg) A "univ env alg" of $\mathfrak{g}$ is a pair $(\mathcal{U}(\mathfrak{g}), \sigma)$ w/

1. $\mathcal{U}(\mathfrak{g})$ a unital assoc alg
2. $\sigma : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g})$ a Lie alg hom

satisfying the universal property: $\forall$ unital assoc alg A, TFDC

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\forall \text{ Lie alg hom}"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowright"] & A \\
\mathcal{U}(\mathfrak{g}) \arrow[ur, dashed, "\exists! \text{ unital assoc alg hom}"']
\end{tikzcd}$$

A & $\mathcal{U}(\mathfrak{g})$ equipped w/ commutator Lie bracket

Remarks

1. Uniqueness of $\mathcal{U}(\mathfrak{g})$ follows from usual univ prop uniqueness argument
2. σ is, in fact, an embedding by Poincaré–Birkhoff–Witt thm
3. From $A = \mathrm{End}(V)$, we see any rep (π, V) of $\mathfrak{g}$ factors through a rep $\tilde{\pi} : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathrm{End}(V)$

   for matrix Lie groups, $\tilde{\pi} \equiv$ matrix-vec mult

Thm (Existence of $\mathcal{U}(\mathfrak{g})$) $(\mathcal{U}(\mathfrak{g}), \sigma)$ exists for all Lie algebras $\mathfrak{g}$

Proof (By construction)

Form the tensor algebra $T(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$ ← graded by n

1. Lifting $[-,-]_{\mathfrak{g}}$ to $T(\mathfrak{g})$ Define $[-,-]$ recursively grade-by-grade

   – base case define $[-,-] : \mathfrak{g}^{\otimes 2} \longrightarrow \mathfrak{g}$ by

   $$[a,b] := a \otimes b - b \otimes a \equiv [a,b]_{\mathfrak{g}}$$

– recursive step: define $[-,-] : \mathfrak{g}^{\otimes m} \longrightarrow \mathfrak{g}^{\otimes m-1}$ by

$$\left.\begin{array}{l}[a\otimes b, c] := a\otimes[b,c] + [a,b]\otimes c \\ [a, b\otimes c] := [a,b]\otimes c + b\otimes[a,b]\end{array}\right\}$$ ie. define $[-,-]$ to be a derivation wrt grading

We can check that $[-,-]$ is bilinear, skew-sym, + satisfies Jacobi identity

Remark $(T(\mathfrak{g}), \otimes, [-,-])$ forms a "Poisson alg"

② Forming $\mathcal{U}(\mathfrak{g})$ Define ideal $J := \langle x\otimes y - y\otimes x - [x,y] \rangle$, + form the quotient $\mathcal{U}(\mathfrak{g}) := T(\mathfrak{g})/J$; finally let

$\sigma : \mathfrak{g} \longrightarrow \mathcal{U}(\mathfrak{g}) : x \longmapsto x + J.$ ← quotient map

Then $\mathcal{U}(\mathfrak{g})$ is a unital assoc alg, + σ is a Lie alg hom.

③ Checking univ prop Given $f : \mathfrak{g} \to A$, use univ prop of $T(\mathfrak{g})$ to get $\tilde{f} : T(\mathfrak{g}) \longrightarrow A$, then show that $\tilde{f}$ factors through $\hat{f} : \mathcal{U}(\mathfrak{g}) \longrightarrow A$ by univ prop of quotient.

For uniqueness note that $T(\mathfrak{g}) = \langle 1, \mathfrak{g} \rangle$, so $\mathcal{U}(\mathfrak{g}) = \langle 1, \sigma(\mathfrak{g}) \rangle$. ▨

Aside We can alternately view $\mathfrak{g} = T_1(G)$ as the set $\mathcal{V}(G)^e$ of left-invariant vector fields.

ie. invariant under $(g\cdot -)_* : TG \longrightarrow TG$ for each $g \in G$

So $\mathfrak{g} \cong \mathcal{V}(G)^e = \left\{ \begin{array}{l} 1^{st} \text{ order left-invariant} \\ \text{differential ops on } C^\infty(G) \end{array} \right\}$.

Then it turns out that† Helgason (2021) ch 2 thm 1.9

$$\mathcal{U}(\mathfrak{g}) \cong \left\{ \begin{array}{l} \text{(all) left-invariant} \\ \text{diff ops on } C^\infty(G) \end{array} \right\} = \langle id, \mathcal{V}(G)^e \rangle.$$ as alg of ops on $C^\infty(G)$

§2. Hopf Algebras

† Turaev (2016) ch 11

Def (Hopf alg) A "Hopf alg"[†] is a tuple $(A, \Delta, \varepsilon, s)$, w/

① A a unital alg ← unit 1_A & product $m: A \times A \longrightarrow A$

② $\Delta: A \longrightarrow A \otimes A$ an alg homomorphism ("coproduct")

③ $\varepsilon: A \longrightarrow \mathbb{F}$ an alg homomorphism ("counit")

④ $s: A \longrightarrow A$ a linear map ("antipode")

satisfying

① $(id_A \otimes \Delta) \circ \Delta = (\Delta \otimes id_A) \circ \Delta$ ("coassoc")

② $(\varepsilon \otimes id_A) \circ \Delta = (id_A \otimes \varepsilon) \circ \Delta = id_A$ ("counit")

③ $m \circ (s \otimes id_A) \circ \Delta = m \circ (id_A \otimes s) \circ \Delta = \varepsilon \cdot 1_A$ ("antipode")

Remark This def is succinct but hides some symmetry. We can arrive at a nicer, more symmetric def at the expense of some extra work

Def (Monoid) Given a monoidal cat $(\mathcal{C}, \otimes, I)$, a "monoid in $\mathcal{C}$" comprises (M, m, η), w/

unit object $I \in \mathcal{C}$

① $M \in \mathcal{C}$

② $m: M \otimes M \longrightarrow M$ ("product")

③ $\eta: I \longrightarrow M$ ("unit")

st TFDC:

associator

$$\begin{tikzcd}[column sep=small]
(M\otimes M)\otimes M \arrow[rr, "\cong"] \arrow[d, "m\otimes id"'] & & M\otimes(M\otimes M) \arrow[d, "id\otimes m"] \\
M\otimes M \arrow[dr, "m"'] & & M\otimes M \arrow[dl, "m"] \\
& M &
\end{tikzcd}$$

("assoc")

$$\begin{tikzcd}
I\otimes M \arrow[r, "\eta\otimes id"] \arrow[dr, "\cong"'] & M\otimes M \arrow[d, "m"] & M\otimes I \arrow[l, "id\otimes\eta"'] \arrow[dl, "\cong"] \\
& M &
\end{tikzcd}$$

left unitor

right unitor

("left + right unit laws")

Def (Monoid morphism) A morphism $(M,m,\eta) \xrightarrow{f} (M',m',\eta')$ is a morphism $M \xrightarrow{f} M$ st TFDC

$$\begin{tikzcd}
M\otimes M \arrow[r, "f\otimes f"] \arrow[d, "m"'] & M'\otimes M' \arrow[d, "m'"] \\
M \arrow[r, "f"'] & M'
\end{tikzcd}$$

("multiplicativity")

$$\begin{tikzcd}
M \arrow[rr, "f"] & & M' \\
& I \arrow[ul, "\eta"] \arrow[ur, "\eta'"'] &
\end{tikzcd}$$

("unit-preserving")

Def (cat of (co)monoids) Given monoidal cat $(C,\otimes,I)$, form the cat of (co)monoids in C, $(Co)MonC$, w/ morphisms of monoids as above.

† Joyal & Street (1993)

Thm Given symmetric monoidal cat $(\mathcal{C},\otimes,I)$, $(\mathrm{Co})\mathrm{Mon}\,\mathcal{C}$ can be upgraded to a (symmetric) monoidal cat.
In particular, if (M,m,η), (M',m',η') are monoids in $(\mathcal{C},\otimes,I)$, then the $M\otimes M'$ is considered a monoid via the data

$(M\otimes M',\ (m\otimes m')\circ(\mathrm{id}\otimes\tau\otimes\mathrm{id}),\ \eta\otimes\eta')$, for

ie. $(a\otimes a')(b\otimes b') \equiv ab\otimes a'b'$, writing mult as juxtaposition

$\tau : M'\otimes M \longrightarrow M\otimes M' : a\otimes b \longmapsto b\otimes a$.

Def (Bimonoid) Given symmetric monoidal cat $(\mathcal{C},\otimes,I)$, the cat of "bimonoids in $\mathcal{C}$" is

$$\mathrm{BiMon}\,\mathcal{C} := \mathrm{Mon}\,\mathrm{CoMon}\,\mathcal{C} \cong \mathrm{CoMon}\,\mathrm{Mon}\,\mathcal{C}$$

† R. Street (2007)

Fact $(\mathrm{Vect}_{\mathbb{F}},\otimes,\mathbb{F})$ is a symmetric monoidal cat.

Def ((Co)alg) A "(co)algebra" is a (co)monoid in Vect.

Def (Bialg) A "bialgebra" is a bimonoid in Vect.

Notation We usually give a bialg as a pentuple $(A,m,\eta,\Delta,\varepsilon)$, w/
1. (A,m,η) forming an alg
2. (A,Δ,ε) forming a coalg

compat conditions % (m,η) and (Δ,ε) follow by each pair being homomorphism wrt the other

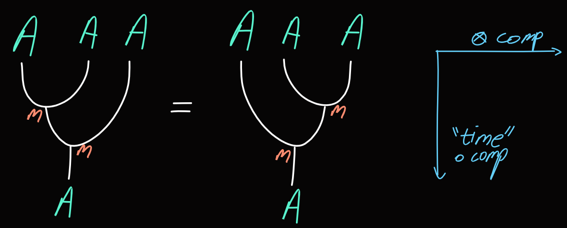
Def (Hopf alg) A "Hopf alg" is a sextuple $(A,m,\eta,\Delta,\varepsilon,S)$, comprising a bialg + a linear map $S : A\longrightarrow A$ st TFDC

"antipode"

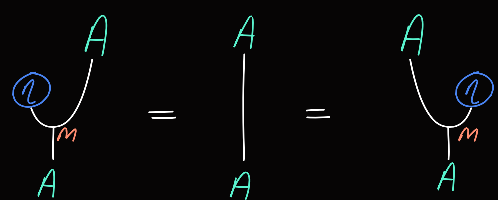
$$\begin{tikzcd}
 & A\otimes A \arrow[r, "id\otimes S"] \arrow[d, phantom, "\circlearrowright"] & A\otimes A \arrow[dr, "m"] & \\
A \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[r, "\varepsilon"] & \mathbb{F} \arrow[rr, "\eta"] & & A \\
 & A\otimes A \arrow[r, "S\otimes id"'] \arrow[u, phantom, "\circlearrowright"] & A\otimes A \arrow[ur, "m"'] &
\end{tikzcd}$$

† Heunen et al (2013) §4.3

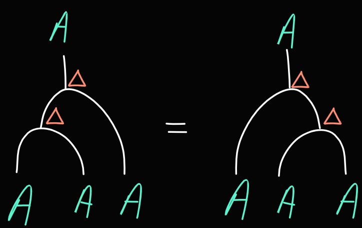
Remark (Diagrammatic calculus) Let us expand the def of a Hopf monoid $(A, m, \eta, \Delta, \varepsilon, S)$ in a general braided monoidal cat ito string diagrams

① 

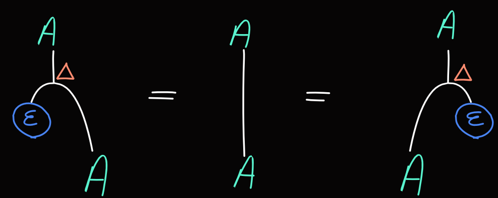
assoc
"$(ab)c = a(bc)$"

② 

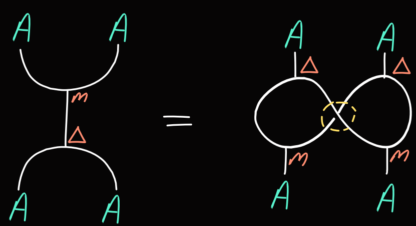
unit laws
"$1a = a = a1$"

③ 

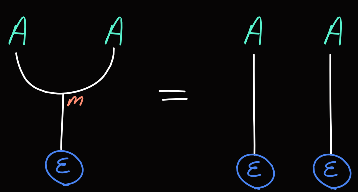
coassoc

④ 

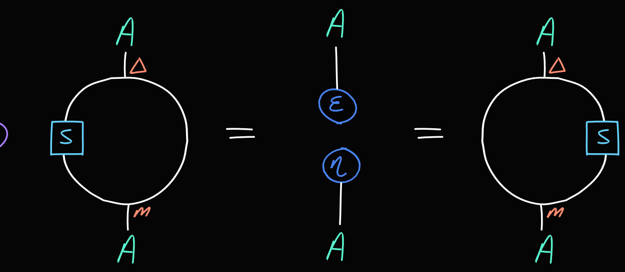
counit laws

⑤ 

multiplicativity of Δ
"$\Delta(ab) = \Delta(a)\Delta(b)$"

⑥ 

multiplicativity of ε
"$\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$"

⑦ 

antipode law

† Turaev (2016), ch 11

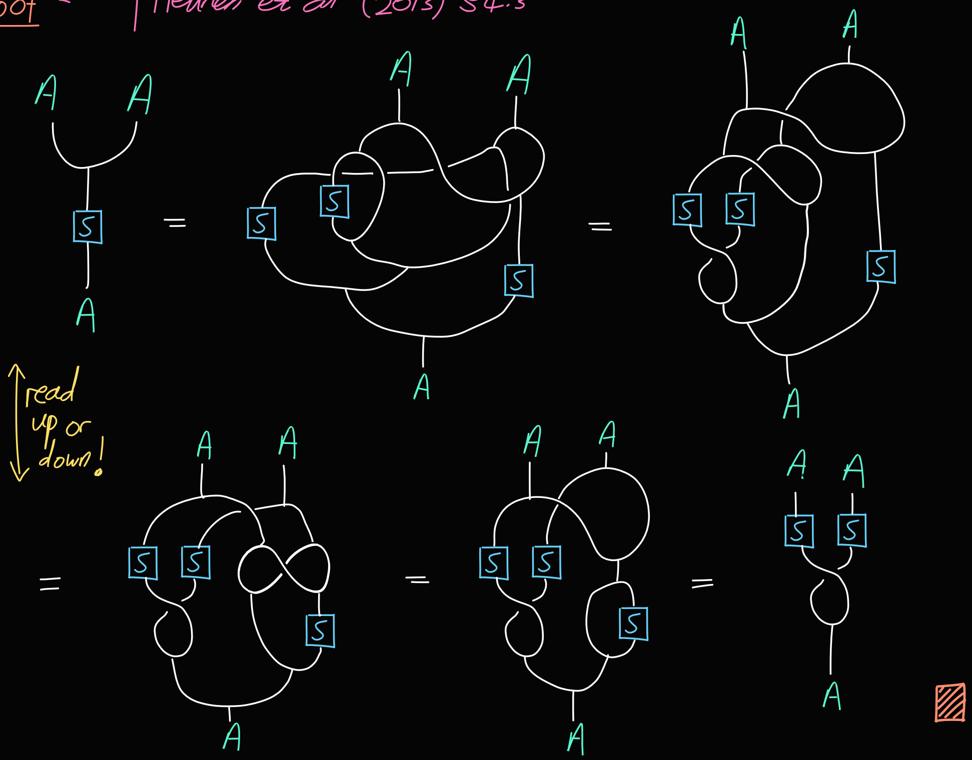
Thm For Hopf alg $(A, m, \eta, \Delta, \varepsilon, s)$, s is an antimorphism of both the alg & coalg structures of A

ie. $m \circ \tau \circ (s \otimes s) = s \circ m$, and

$\Delta \circ s = (s \otimes s) \circ \tau \circ \Delta$, for

$\tau : A \otimes A \longrightarrow A \otimes A : a \otimes b \longmapsto b \otimes a$

Proof ← † Heunen et al (2013) §4.3



Examples

① **Group algebras** For G group, $\mathbb{F}$ field, the group alg $\mathbb{F}[G]$ is a Hopf alg, w/

- $m(g \otimes g') = gg'$ *
- $\Delta(g) = g \otimes g$ *
- $\varepsilon(g) = 1_{\mathbb{F}}$ *

*(extend linearly)

- $\eta(\lambda) = \lambda 1_G$
- $s(g) = g^{-1}$ *

② Univ env algs For Lie alg $\mathfrak{g}$, $\mathcal{U}(\mathfrak{g})$ is a Hopf alg, w/

- $\Delta(X) = X \otimes 1 + 1 \otimes X$ *
- $\varepsilon(X) = 0$ * *(extend linearly)
- $s(X) = -X$ *

§3. First Examples of Quantum Groups

† Gelca (2014) / Turaev (2016)

Recall Quantum groups are Hopf algebras obtained by "1-parameter deformation" of some "classical" Hopf alg.

typically $\mathcal{U}(\mathfrak{g})$ for (simple/semisimple) Lie alg $\mathfrak{g}$, but also others, eg. group alg $\mathbb{C}[G]$ of Lie group

Example ($\mathcal{U}_q(\mathfrak{g})$) Let $\mathfrak{g}$ be a Lie alg of type A, D, or E w/ Cartan matrix (a_{ij}). Recall from Lie theory that these entries, defined

$$a_{ij} := 2\frac{\langle r_i, r_j\rangle}{\langle r_i, r_i\rangle}, \text{ for simple roots } r_i, \ i \in \{1, \dots, m\}, \text{ are st}$$

$a_{ii} = 2$ and $a_{ij} = a_{ji} \in \{0, -1\}$ for $i \neq j$.

Fix parameter $q \in \mathbb{C} \setminus \{-1, 0, 1\}$, + define quantum group $\mathcal{U}_q(\mathfrak{g})$ as the $\mathbb{C}$-alg given by generators E_i, F_i, K_i, K_i^{-1}, w/ rels

① $K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1$

② $K_iE_j = q^{a_{ij}}E_jK_i, \quad K_iF_j = q^{-a_{ij}}F_jK_i$

③ $E_iF_j - F_jE_i = \delta_{ij}\dfrac{K_i - K_i^{-1}}{q - q^{-1}}$

④ $E_iE_j = E_jE_i, \quad F_iF_j = F_jF_i \quad$ if $a_{ij} = 0$

⑤ $\left.\begin{array}{l} E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \\ F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0 \end{array}\right\}$ if $a_{ij} = -1$

Then $U_q(\mathfrak{g})$ is a Hopf alg w/ ops

$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$	$S(E_i) = -K_i^{-1} E_i$	$\varepsilon(E_i) = \varepsilon(F_i) = 0$
$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$	$S(F_i) = -F_i K_i$	$\varepsilon(K_i) = 1$
$\Delta(K_i) = K_i \otimes K_i$	$S(K_i) = K_i^{-1}$	

Moreover, setting $K_i = e^{-hH_i/2}$, $q = e^{-h/2}$ + taking $h \to 0$ recovers the Chevalley gen-rel presentation of $U(\mathfrak{g})$.