Introduction to Quantum Groups II

§3. First Examples of Quantum Groups†

† Gelca (2014) / Turaev (2016)

Recall Quantum groups are Hopf algebras obtained by "1-parameter deformation" of some "classical" Hopf alg.

typically $\mathcal{U}(\mathfrak{g})$ for (simple/semisimple) Lie alg $\mathfrak{g}$, but also others, eg. group alg $\mathbb{C}[G]$ of Lie group

Example ($\mathcal{U}_q(\mathfrak{g})$) Let $\mathfrak{g}$ be a Lie alg of type A, D, or E w/ Cartan matrix (a_{ij}). Recall from Lie theory that these entries, defined

$$a_{ij} := 2\frac{\langle r_i, r_j\rangle}{\langle r_i, r_i\rangle}, \text{ for simple roots } r_i, \ i \in \{1, \dots, m\}, \text{ are st}$$

$$a_{ii} = 2 \quad \text{and} \quad a_{ij} = a_{ji} \in \{0, -1\} \text{ for } i \neq j.$$

Fix parameter $q \in \mathbb{C} \setminus \{-1, 0, 1\}$, + define quantum group $\mathcal{U}_q(\mathfrak{g})$ as the $\mathbb{C}$-alg given by generators E_i, F_i, K_i, K_i^{-1}, w/ rels

1. $K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1$
2. $K_iE_j = q^{a_{ij}}E_jK_i, \quad K_iF_j = q^{-a_{ij}}F_jK_i$
3. $E_iF_j - F_jE_i = \delta_{ij}\dfrac{K_i - K_i^{-1}}{q - q^{-1}}$
4. $E_iE_j = E_jE_i, \quad F_iF_j = F_jF_i \quad$ if $a_{ij} = 0$
5. $E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0$, $\quad F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0 \quad$ if $a_{ij} = -1$

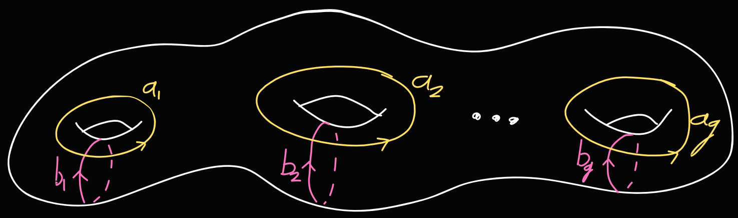
Then $\mathcal{U}_q(\mathfrak{g})$ is a Hopf alg w/ ops

$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i$	$S(E_i) = -K_i^{-1}E_i$	$\varepsilon(E_i) = \varepsilon(F_i) = 0$
$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i$	$S(F_i) = -F_iK_i$	$\varepsilon(K_i) = 1$
$\Delta(K_i) = K_i \otimes K_i$	$S(K_i) = K_i^{-1}$	

Moreover, setting $K_i = e^{-hH_i/2}$, $q = e^{-h/2}$ + taking $h \to 0$ recovers the Chevalley gen-rel presentation of $\mathcal{U}(\mathfrak{g})$.

"Recall" For surface Σ_g, $H_1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^g \times \mathbb{Z}^g$ by choice of "canonical basis".

eg.



$$\sum p_j a_j + q_j b_j \in H_1(\Sigma_g, \mathbb{Z})$$
$$\downarrow$$
$$(p,q) \quad \in \mathbb{Z}^g \times \mathbb{Z}^g$$

Thm Given such a basis, we get a unique basis $\varsigma_1, \ldots, \varsigma_g$ of the space $\mathcal{H}(\Sigma_g)$ of holomorphic 1-forms on Σ_g w/ $\int_{a_j} \varsigma_k = \delta_{jk}$.

Def Define the Jacobian variety $\mathcal{J}(\Sigma_g) = \mathbb{C}^g / \Lambda(\Pi)$ (symp mfd (ie. classical phase space)) w/ "period matrix" Π defined $\int_{b_j} \varsigma_k = \Pi_{jk}$.

Fact As a real space, $\mathcal{J}(\Sigma_g) \cong \mathbb{R}^{2g} / \mathbb{Z}^{2g} \cong H_1(\Sigma_g, \mathbb{R}) / H_1(\Sigma_g, \mathbb{Z})$.
The real coords $(x,y) \in \mathbb{R}^{2g}$ on $\mathcal{J}(\Sigma_g)$ are related to the complex coords $z \in \mathbb{C}^g$ by $z = x + \Pi y$

Def Define the "Heisenberg group w/ integral entries"

$$H(\mathbb{Z}^g) = \{(p,q,k) \mid p,q \in \mathbb{Z}^g, k \in \mathbb{Z}\}, \text{ w/ composition}$$

$$(p,q,k)(p',q',k') = (p+p',\ q+q',\ k+k' + \underbrace{p \cdot q' - p' \cdot q}_{\substack{= \text{intersxn form between} \\ (p,q),(p',q') \in H_1(\Sigma_g, \mathbb{Z})}}).$$

ie. $H(\mathbb{Z}^g)$ = central extension of $H_1(\Sigma_g, \mathbb{Z}) \cong \mathbb{Z}^g \times \mathbb{Z}^g$ by cocycle given by intersxn form

The "finite Heisenberg group" $H(\mathbb{Z}_N^g)$ comprises elements of the form

$$H(\mathbb{Z}_N^g) = \{(p,q,k) \mid p,q \in \mathbb{Z}_N^g, k \in \mathbb{Z}_{2N}\}$$

Fact ① $H_1(\Sigma_g, \mathbb{Z}) \cong \{\text{exp fns on } \mathcal{J}(\Sigma_g)\}$

$\sum p_j a_j + q_j b_j \longmapsto \exp[2\pi i(p\cdot x + q\cdot y)]$

② $\mathbb{C}[H_1(\Sigma_g, \mathbb{Z})] \cong \{\text{trig polynomials on } \mathcal{J}(\Sigma_g)\}$

③ $H(\mathbb{Z}_N^g) \cong \{\text{quantized exp fns on } \mathcal{J}(\Sigma_g)\}$

$(p,q,k) \longmapsto \mathrm{op}\left(\exp\left[2\pi i(p\cdot x + q\cdot y) + \frac{\pi i}{N}k\right]\right)$

Example (Heisenberg group) $A = \mathbb{C}[H_1(\Sigma_g, \mathbb{Z})]$ is our "classical" Hopf alg. Fix parameter h, + let $t = e^{i\pi h}$. Consider free module

$A_h = \mathbb{C}[t, t^{-1}]\, H_1(\Sigma_g, \mathbb{Z})$ w/ product

$(p,q) *_t (p',q') = t^{i\pi(p\cdot q - q\cdot p)}(p+p', q+q')$.

"deformation" of A wrt h

Then it is easily shown that $A_h \cong \mathbb{C}[H(\mathbb{Z}^g)]$.

Formally imposing $t^{2N} = 1$ & $(p,q)^N = (0,0)$ yields $\mathbb{C}[H(\mathbb{Z}_N^g)]$.

there is a broad theory by Kirillov & Reshetikhin for going

quant grp $\rightsquigarrow$ quant grp at root of unity

"quantum group of alg of trig polynomials at root of unity"

§4. Return to Yang-Baxter eq.

† Kassel (1995), ch 8

Motto Quantum groups exist as a machinery to produce solutions to the Yang-Baxter equation. We will now explore how. We will need some more terminology first.

Def (R-matrix) An element $c \in \mathrm{Aut}(V \otimes V)$ (vec space) is an "R-matrix" if it solves the Yang-Baxter eq.

$$(c\otimes id)(id \otimes c)(c \otimes id) = (id\otimes c)(c\otimes id)(id \otimes c).$$

Example Transposition, $\tau: V\otimes V \longrightarrow V\otimes V : a\otimes b \longmapsto b\otimes a$ is an R-matrix, since $(12)(23)(12) = (23)(12)(23)$ in S_3.

Thm If c is an R-matrix, then so are

① λc, (scalar) ② c^{-1}, ③ $\tau\circ c\circ\tau$.

Def (Quasi-cocommutative bialg) A bialg $(A, m, \eta, \Delta, \varepsilon)$ is "quasi-cocommutative" if $\exists$ invertible $R\in A\otimes A$ st $\Delta^{op}(-) = R\,\Delta(-)\,R^{-1}$.

opposite coproduct $\Delta^{op} = \tau\circ\Delta$

Then R is called the "universal R-matrix".

Remark A cocommutative bialg is quasi-cocommutative w/ $R = 1\otimes 1$.
The noncocommutativity of a quasi-cocommutative bialg is thus controlled by R.

Notation For alg A, $X = x_1\otimes\cdots\otimes x_n \in A^{\otimes n}$, and $(k_1,\dots,k_n)$ a tuple in $\{1,\dots,p\}$ w/ $p\geq n$, write

$$X_{k_1\dots k_n} = y_1\otimes\cdots\otimes y_p \in A^{\otimes p},$$

setting each $y_{k_j} = x_j$, and $y_k = 1$ otherwise.

Extend notation linearly to non-pure tensors.

eg. if $X = a_j\otimes b_j$, then $X_{31} = b_j\otimes 1\otimes a_j$

Def (Braided bialg) A quasi-cocommutative bialg $(A, m, \eta, \Delta, \varepsilon, R)$ is "braided" if its univ R-matrix satisfies

① $(\Delta\otimes id)(R) = R_{13}R_{23}$

② $(id\otimes\Delta)(R) = R_{13}R_{12}$

aka. "quasitriangular", (but we prefer braided ∵ their rep cat is braided)

Remark The terms "(quasi)-cocommutative" & "braided" apply just so to Hopf algs (w/out additional conditions)

Examples (Braided bialgs)

① Trivial example: Any cocommutative bialg is braided with $R = 1 \otimes 1$

② Sweedler's 4D alg: H generated by x, y, w/ rels

$$x^2 = 1, \quad y^2 = 0, \quad yx + xy = 0,$$

is a Hopf alg w/

$$\Delta(x) = x \otimes x \qquad \Delta(y) = 1 \otimes y + y \otimes x$$
$$\varepsilon(x) = 1 \qquad \varepsilon(y) = 0$$
$$S(x) = x \qquad S(y) = xy,$$

+ is braided w/

$$R = \tfrac{1}{2}(1 \otimes 1 + 1 \otimes x + x \otimes 1 - x \otimes x) + \tfrac{\lambda}{2}(y \otimes y + y \otimes xy + xy \otimes xy - xy \otimes y).$$

Thm If $(A, m, \eta, \Delta, \varepsilon, R)$ is a braided bialg, represented on vec space V, then $c = \tau \circ R \in \mathrm{Aut}(V \otimes V)$ is an R-matrix

easily generalized to a linear map $c_{V,W} : V \otimes W \to W \otimes V$ for 2 reps, V & W

Proof

① Rewriting Yang-Baxter eq: In this case, the Yang-Baxter becomes

$$(c \otimes id)(id \otimes c)(c \otimes id) = (id \otimes c)(c \otimes id)(id \otimes c)$$

$$\| \qquad\qquad\qquad \|$$

$$(\tau \otimes id)(R \otimes id)(id \otimes \tau)(id \otimes R)(\tau \otimes id)(R \otimes id) \qquad (id \otimes \tau)(id \otimes R)(\tau \otimes id)(R \otimes id)(\tau \otimes id)(id \otimes R)$$

$$\| \qquad\qquad\qquad \|$$

$$(\tau \otimes id) R_{13} R_{13} R_{12} \qquad (id \otimes \tau) R_{13} R_{13} R_{23}$$

$$\| \qquad\qquad\qquad \|$$

$$\boxed{R_{23} R_{13} R_{12} = R_{12} R_{13} R_{23}} \quad (*)$$

② Checking Yang-Baxter eq: We check $(*)$ using defining properties of R.

Let $R = \alpha_j \otimes \beta_j$ (implied sum). Then

$$R_{12} R_{13} R_{23} R_{12}^{-1} = R_{12} [(\Delta \otimes id)(R)] R_{12}^{-1} \quad \text{(braiding)}$$
$$= R_{12} [\Delta(\alpha_j) \otimes \beta_j] R_{12}^{-1} \quad \text{(def of } R\text{)}$$

$$= R_{12}\Delta(\alpha_j)R_{12}^{-1}\otimes\beta_j \qquad \text{(def of } R_{12})$$
$$= (\tau\circ\Delta)(\alpha_j)\otimes\beta_j \qquad \text{(quasi-cocomm.)}$$
$$= ((\tau\circ\Delta)\otimes id)(R) \qquad \text{(rearrangement)}$$
$$= \tau_{12}[(\Delta\otimes id)(R)] \qquad \text{(def of } \tau_{12})$$
$$= \tau_{12}(R_{13}R_{23}) \qquad \text{(braiding)}$$
$$= R_{23}R_{13} \qquad \text{(def of } \tau_{12}) \quad \blacksquare$$

Thm (Properties of R) In braided bialg $(A, m, \eta, \Delta, \varepsilon, H)$, we have

① $(\varepsilon\otimes id)(R) = 1 = (id\otimes\varepsilon)(R)$

If we further have antipode S, then

② $(S\otimes id)(R) = R^{-1} = (id\otimes S^{-1})(R)$

③ $(S\otimes S)(R) = R$

§5. Square of the Antipode

† Kassel (1995) ch 8, Gelaca (2014) §8.1.3

Recall from our examples of Hopf algs so far,

$\mathbb{F}[G]$ – $S(g) = g^{-1}$
$\mathcal{U}(\mathfrak{g})$ – $S(x) = -x$

We may wonder, does $S^2 = 1$ generally?

i.e. $\Delta^{op} \equiv \tau\Delta = \Delta$
or $R = 1\otimes 1$

Thm In a cocommutative Hopf alg $(A, m, \eta, \Delta, \varepsilon, S, R = 1\otimes 1)$, $S^2 = id$

Proof (follows from next thm) $\blacksquare$

Notation (Sweedler's notation) For $a\in A$, we have $\Delta(a) = \sum_j a_j'\otimes a_j''$. Write instead $\Delta(a) = \sum_{(a)} a'\otimes a''$ to save indices.

Coassoc of Δ requires

$$\sum_{(a)} \Big(\sum_{(a')} (a')' \otimes (a')'' \Big) \otimes a'' = \sum_{(a)} a' \otimes \Big(\sum_{(a'')} (a'')' \otimes (a'')'' \Big).$$

We write both sides simply as $\sum_{(a)} a' \otimes a'' \otimes a'''$.

Def For quasi-cocommutative Hopf alg $(A, m, \eta, \Delta, \varepsilon, S, R)$ w/ $R = \sum_j \alpha_j \otimes \beta_j$,

let $u \equiv m\big[(S \otimes id)(\tau R)\big] = \sum_j S(\beta_j)\alpha_j \quad \in A$

Thm For u as above, and $R = \sum_j \alpha_j \otimes \beta_j$,

① $u^{-1} = \sum_j S^{-1}(\beta_j) S(\alpha_j) = \sum_j \beta_j S^2(\alpha_j)$

② $S^2(-) = u(-)u^{-1}$ ← in particular, if $R = 1 \otimes 1$, then $u = 1$, so $S^2 = id$

Proof

② For $x \in A \otimes A$, we have

$(\Delta^{op} \otimes id)(x)(R \otimes id) = (R \otimes id)(\Delta \otimes id)(x)$. (follows by quasi-cocomm, $\Delta^{op}(-) = R(-)R^{-1}$)

substituting $x = \Delta(a)$,

$$\sum_{j,(a)} a'' \alpha_j \otimes a' \beta_j \otimes a''' = \sum_{j,(a)} \alpha_j a' \otimes \beta_j a'' \otimes a'''.$$

Apply $[m(id \otimes m)] \circ (id \otimes S \otimes S^2) : A^{\otimes 3} \longrightarrow A$ to both sides:

(#) $$\sum_{j,(a)} S^2(a''') S(\beta_j) S(a') a'' \alpha_j = \sum_{j,(a)} S^2(a''') S(a'') S(\beta_j) \alpha_j a'.$$

$S(a'\beta_j) = S(\beta_j)S(a')$, as S is an antiautomorphism

Now by def of antipode,

$$\sum_{(a)} S(a')a'' \otimes a''' = \sum_{(a)} \varepsilon(a')\, 1 \otimes a'' = 1 \otimes a,$$

so $\sum_{(a)} S(a')a'' \otimes S^2(a''') = 1 \otimes S^2(a)$.

Multiplying by $\sum_j \alpha_j \otimes S(\beta_j)$, + applying m, we get the LHS of (#)

$$\sum_{j,(a)} S^2(a''') \, S(\beta_j) \, S(a') \, a'' \alpha_j = \sum_j S^2(a) S(\beta_j) \alpha_j = S^2(a) u .$$

Similarly, for the RHS, first note

$$\sum_{(a)} a' \otimes S(a'' S(a''')) = \sum_{(a)} a' \otimes S(\varepsilon(a'') \, 1) = \sum_{(a)} a' \varepsilon(a'') \otimes S(1) = a \otimes 1,$$

whence (multiplying by $u \otimes 1$, then applying $m\tau$), the RHS of (#) is

$$\sum_{j,(a)} S^2(a''') \, S(a'') \, S(\beta_j) \, \alpha_j a' = ua.$$

So (#) reads $S^2(a) u = ua$

① (skipped) ∎

Cor ① $S^2(u) = u, \quad S^2(u^{-1}) = u^{-1}$

② $uS(u) = S(u)u \in Z(A)$

Thm ① $\varepsilon(u) = 1$

② $\Delta(u) = \underbrace{(R_{21}R)^{-1}}_{=\tau(R)} [u \otimes u] = [u \otimes u] (R_{21}R)^{-1}$

③ $\Delta(S(u)) = (R_{21}R)^{-1} [S(u) \otimes S(u)] = [S(u) \otimes S(u)] (R_{21}R)^{-1}$

④ $\Delta(uS(u)) = (R_{21}R)^{-2} [uS(u) \otimes uS(u)] = [uS(u) \otimes uS(u)] (R_{21}R)^{-2}$

§6. Ribbon Categories

† Bakalov + Kirilov ch 2

Overview "Ribbon Hopf algs" yield link invariants because their rep cat is a "ribbon cat," whose graphical calculus presents morphisms as framed tangles.

In fact, any ribbon cat yields link invariants.

Hence, we will define "ribbon" twice, starting at the level of cats, where we first must define "rigid" + "balanced".

Step 1 is to discuss the idea of "duality" in a monoidal cat.

Def (Dual objects) Given $V \in ob(\mathcal{C})$ in monoidal cat $(\mathcal{C}, \otimes, I)$, a "right dual" to V is an object V^* w/ morphisms

$$e_V : V^* \otimes V \longrightarrow I$$
$$i_V : I \longrightarrow V \otimes V^*$$

st TFDC

$$\begin{tikzcd} V \otimes V^* \otimes V \arrow[r, "id \otimes e_V"] & V \\ V \arrow[u, "i_V \otimes id"] \arrow[ur, "id"'] \arrow[u, phantom, "\circlearrowright", shift left=20] \end{tikzcd} \qquad \begin{tikzcd} V^* \otimes V \otimes V^* \arrow[r, "e_V \otimes id"] & V^* \\ V^* \arrow[u, "id \otimes i_V"] \arrow[ur, "id"'] \arrow[u, phantom, "\circlearrowright", shift left=20] \end{tikzcd}$$

suppress $V \cong I \otimes V$

"rigidity axioms"

A "left dual" of V is an object ${}^*V$, w/ morphisms

$$e'_V : V \otimes {}^*V \longrightarrow I$$
$$i'_V : I \longrightarrow {}^*V \otimes V$$

satisfying similar rigidity axioms.

Def (Rigid monoidal cat) A monoidal cat $(\mathcal{C}, \otimes, I)$ is "rigid" if all objects have left + right duals.

Examples (Rigid cats)

① $(\mathrm{Vect}_{\mathbb{F}}, \otimes, \mathbb{F})$ is rigid wrt usual dual

② $(\mathrm{Rep}(A), \otimes, \mathbb{F})$, the cat of finite-dim reps of Hopf alg A is rigid. (tensor prod)

We must first understand how $\mathrm{Rep}(A)$ is monoidal.

- Given $V, W \in ob(\mathrm{Rep}(A))$, $V \otimes W \in ob(\mathrm{Rep}(A))$, via
$$a \cdot (v \otimes w) := \Delta(a)\, v \otimes w$$
- $\mathbb{F} \in ob(\mathrm{Rep}(A))$ via
$$a \cdot 1 := \varepsilon(a)\, 1$$

Then the duals are the usual vect space duals w/ axn
$$(a \cdot v^*)(w) := v^*(Sw).$$

Thm The duality in a rigid monoidal cat $(\mathcal{C}, \otimes, I)$ is compatible w/ the monoidal structure insofar as

① $I^* = I = {}^*I$
② $(V \otimes W)^* = W^* \otimes V^*$
} up to unique iso

③ As a functor $\mathcal{C} \longrightarrow \mathcal{C}^{op}$, the right dual respects the associator of $(\mathcal{C}, \otimes, I)$, + any braiding it may have:

$$\alpha_{uvw}^* = \alpha_{w^* v^* u^*}$$

$$\tau_{uv}^* = \tau_{u^* v^*}$$

Def (Ribbon cat) A "ribbon cat" (aka "balanced rigid braided monoidal cat") is a rigid braided monoidal cat $(\mathcal{C}, \otimes, I, \tau)$ equipped w/ a "twist", ie. a nat trans $\theta : id_{\mathcal{C}} \Longrightarrow id_{\mathcal{C}}$ w/ components $\theta_V : V \xrightarrow{\sim} V$ satisfying

① $\theta_{V \otimes W} = \tau_{WV} \tau_{VW} (\theta_V \otimes \theta_W)$
② $\theta_I = id$
③ $\theta_{V^*} = (\theta_V)^*$
} "balancing axioms"

Lem A twist θ in a rigid braided monoidal cat $(\mathcal{C}, \otimes, I, \tau, *)$ yields a nat trans $\delta : id_{\mathcal{C}} \Longrightarrow (-)^{**}$ w/ components $\delta_V : V \xrightarrow{\sim} V^{**}$ st

① $\delta_{V \otimes W} = \delta_V \otimes \delta_W$
② $\delta_I = id$
③ $\delta_{V^*} = (\delta_V^*)^{-1}$

+ vice-versa.

δ is called a "pivotal structure"

ie. δ and θ are equivalent data