# Introduction to Quantum Groups III

## Reminder

**Def** (Rigid monoidal cat) A monoidal cat $(\mathcal{C}, \otimes, I)$ is "rigid" if all objects have left + right duals.

aka "autonomous"

**Def** (Ribbon cat) A "ribbon cat" is a rigid braided monoidal cat $(\mathcal{C}, \otimes, I, \tau)$ equipped w/ a "twist", ie. a nat trans $\theta : id_{\mathcal{C}} \Rightarrow id_{\mathcal{C}}$ w/ components $\theta_V : V \xrightarrow{\sim} V$ satisfying

aka "balanced rigid braided monoidal cat"

$$\left.\begin{array}{ll} ① & \theta_{V \otimes W} = \tau_{WV}\tau_{VW}(\theta_V \otimes \theta_W) \\ ② & \theta_I = id \\ ③ & \theta_{V^*} = (\theta_V)^* \end{array}\right\} \text{"balancing axioms"}$$

**Lem** A twist $\theta$ on a rigid braided monoidal cat $(\mathcal{C}, \otimes, I, \tau)$ yields a nat trans $\delta : id_{\mathcal{C}} \Rightarrow (-)^{**}$ w/ components $\delta_V : V \xrightarrow{\sim} V^{**}$ st

① $\delta_{V \otimes W} = \delta_V \otimes \delta_W$

② $\delta_I = id$

③ $\delta_{V^*} = (\delta_V^*)^{-1}$

+ vice-versa.

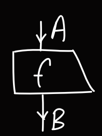
$\delta$ is called a "pivotal structure"

ie. $\delta$ and $\theta$ are equivalent data

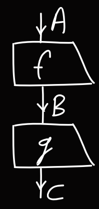
## §7. From Categories to Links

### Lightning Tour (Graphical languages) + Selinger (2009)

① Category

| | | |
|---|---|---|
| $A$ |  | object |
| $A \xrightarrow{f} B$ |  | morphism |

$f \circ g$

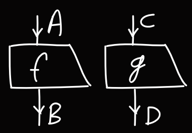



vertical comp

## ② Monoidal Category

$A \otimes B$

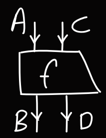

$f \otimes g$




horizontal comp

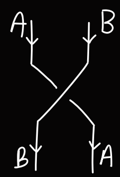
$I$ (empty) unit object

$A \otimes C \xrightarrow{f} B \otimes D$




## ③ Braided Monoidal Category

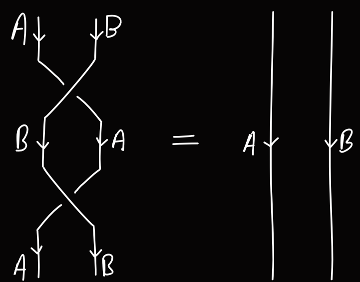
$A \otimes B \xrightarrow{\tau_{AB}} B \otimes A$



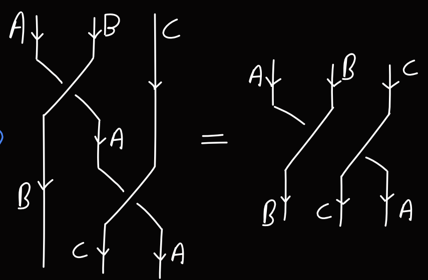

braiding

### Some equations

①

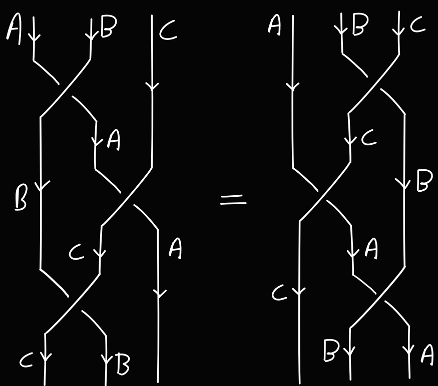



②




hexagon axiom

③




Yang-Baxter

## ④ Balanced Braided Monoidal Category

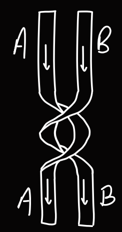
$A$


object

$A \xrightarrow{\theta_A} A$

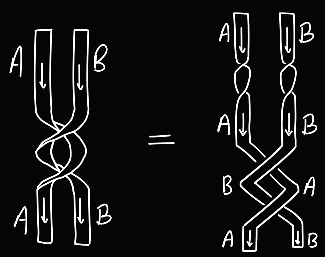

twist

$A\otimes B \xrightarrow{\theta_{A\otimes B}} B\otimes A$




### Some equations

①




balancing axiom

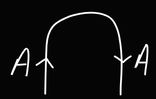
$\theta_{v\otimes w} = \tau_{wv}\tau_{vw}(\theta_v \otimes \theta_w)$

## ⑤ Rigid Monoidal Category

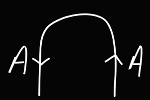
$A^*, {}^*A$


duals

$I \xrightarrow{i_A'} A^*\otimes A$

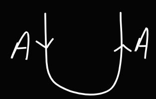



$I \xrightarrow{i_A} A\otimes {}^*A$

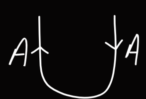



caps

$A\otimes A^* \xrightarrow{e_A'} I$

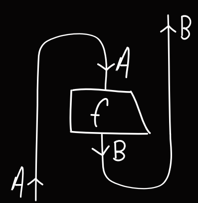



${}^*A\otimes A \xrightarrow{e_A} I$



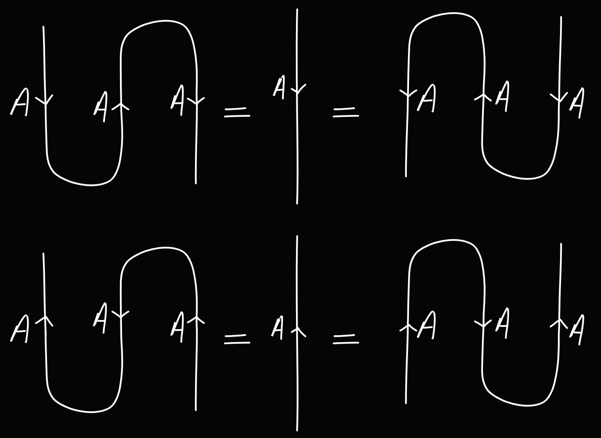

cups

$f^*$




adjoints

### Some equations

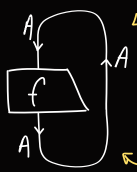
① 
②

yanking identities

## a remark

some apparently-innocent diagrams are not actually well-formed,

≡ trace, Tr(f)

eg.  (given $A \xrightarrow{f} A$) ;

requires morphism $A^{**} \longrightarrow A$

this motivates:

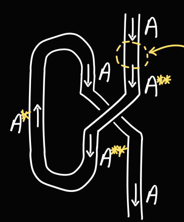
## ⑥ Pivotal Rigid Monoidal Category

$A \xrightarrow{\delta_A} A^{**}$ 

## a remark

we have seen that, in a rigid braided monoidal cat,
twist $\theta$ $\longleftrightarrow$ pivotal structure $\delta$ ;

eg. given $\delta$, define $\theta$ by

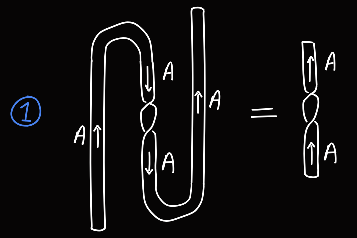
$A \xrightarrow{\theta_A} A$ $\equiv$ 

$\delta_A$ is invisible

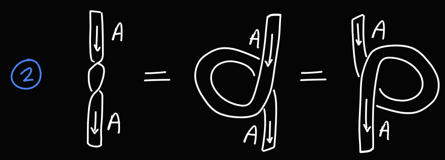
## ⑦ Pivotal/Balanced Rigid Braided Monoidal Category ≡ "Ribbon cat"

aka "tortile" cat

Some equations



balancing axiom
$\theta_{A^*} = (\theta_A)^*$



relationship w/
$\theta$ & $\delta$

**Thm?** (Coherence for ribbon cats) An equation relating ribbon cat morphisms follows from the axioms iff it holds in the graphical calculus, up to 3D framed isotopy

† special case proven in Shum (1994); I can't find general case proof

**Convention** Hereafter, adopt blackboard framing, + draw framed links single-stranded

## §8. From Hopf Algebras to Links

† Kassel (1995), ch 13-14, Gelca (2014) §8.1.4

**Def** (Ribbon Hopf alg) A "ribbon Hopf alg" is a braided Hopf alg $(A, m, \eta, \Delta, \varepsilon, S, R)$ w/ some $\theta \in Z(A)$ st

1. $\theta^2 = u\,S(u)$
2. $\Delta(\theta) = (R_{21}R)^{-1}[\theta \otimes \theta]$
3. $S(\theta) = \theta$
4. $\varepsilon(\theta) = 1$

$uS(u) \in Z(A)$ satisfies these, so we just require a central square root preserving these

**Thm** (Rep cats of algebras) The following table gives the correspondence between algs + their rep cats:

| Algebra $A$ $\Rightarrow$ | Cat $\mathrm{Rep}(A)$ |
|---|---|
| bialg | monoidal |
| Hopf alg | rigid monoidal |
| braided bialg | braided monoidal |
| braided Hopf alg | rigid braided monoidal |
| ribbon Hopf alg | ribbon |

## Proof Sketch

① **bialg** We saw last time that bialg $(A, m, \eta, \Delta, \varepsilon)$ yields monoidal cat $(\mathrm{Rep}(A), \overset{\text{tensor prod}}{\otimes}, \mathbb{F})$ as follows

- Given $V, W \in ob(\mathrm{Rep}(A))$, $V \otimes W \in ob(\mathrm{Rep}(A))$, via

  $a \cdot (v \otimes w) := \Delta(a)\, v \otimes w$

- $\mathbb{F} \in ob(\mathrm{Rep}(A))$ via

  $a \cdot 1 := \varepsilon(a)\, 1$

② **Hopf alg** We also saw that for Hopf alg $(A, m, \eta, \Delta, \varepsilon, S)$, $\mathrm{Rep}(A)$ is rigid w/ normal vec space duals + axn

$$(\underbrace{a \cdot v^*}_{\text{dual vec}})(\underbrace{w}_{\text{vec}}) := v^*(Sw)$$

③ **braided bialg** Recall, for braided bialg $(A, m, \eta, \Delta, \varepsilon, R)$, represented on $V, W$, we defined $c_{VW} = \tau_{VW} \circ R : V \otimes W \longrightarrow W \otimes V$, + showed that these satisfy the Yang-Baxter eq. To prove that the $c_{VW}$s are (components of) a braiding on $\mathrm{Rep}(A)$, it remains to check that they are A-linear automorphisms:

$a \in A,\ s \in V,\ t \in W$

$$\begin{aligned}
c_{VW}(a \cdot (s \otimes t)) &= \tau_{VW}(R\,\Delta(a)(s \otimes t)) && \text{(defs of } c \text{ \& A-axn on } V \otimes W) \\
&= \tau_{VW}(\Delta^{op}(a)\,R(s \otimes t)) && \text{(axiom of quasicocom bialg)} \\
&= \Delta(a)\,\tau_{VW}(R(s \otimes t)) && \text{(def of } \Delta^{op}) \\
&= a \cdot (c_{VW}(s \otimes t)). && \text{(defs of } c \text{ \& A-axn on } V \otimes W)
\end{aligned}$$

Finally, the hexagon identity follows from Yang-Baxter, together w/ the easily verified identities,

$$c_{U, V \otimes W} = (\mathrm{id} \otimes c_{UW})(c_{UV} \otimes \mathrm{id})$$
$$c_{U \otimes V, W} = (c_{UW} \otimes \mathrm{id})(\mathrm{id} \otimes c_{VW})$$

④ **braided Hopf alg** (combine ② + ③)

⑤ **ribbon Hopf alg** We need only equip $\mathrm{Rep}(A)$ w/ a twist $\theta_V : V \longrightarrow V$; define $\theta_V(w) = \theta^{-1} \cdot w$.

Then $\theta_V$ is an A-linear automorphism because $\theta \in A$ is central + invertible. Now, we check the balancing axioms,

$s \in V,\ t \in W$

$$(\theta_V \otimes \theta_W)\, c_{WV} c_{VW}\, (s \otimes t)$$

$$= (\theta^{-1} \otimes \theta^{-1})(R_{21} R)(s \otimes t) \quad \text{(defs of twist } \theta \text{ \& braiding } c\text{)}$$

$$= \Delta(\theta^{-1})(s \otimes t) \quad \text{(ribbon Hopf alg axiom)}$$

$$= \theta_{V \otimes W}(s \otimes t), \quad \text{(def of } A\text{-axn on } V \otimes W\text{)}$$

$\alpha \in V^*,\ t \in V$

$$(\theta_V)^*(\alpha)(t) = \alpha(\theta_V(t)) \quad \text{(def of dual map } (\theta_V)^*\text{)}$$

$$= \alpha(\theta^{-1} t) \quad \text{(def of twist } \theta\text{)}$$

$$= \alpha(S(\theta^{-1}) t) \quad \text{(ribbon Hopf alg axiom)}$$

$$= (\theta^{-1} \alpha)(t) \quad \text{(def of axn on } V^*\text{)}$$

$$= [\theta_{V^*}(\alpha)](t) \quad \text{(def of twist } \theta\text{)} \quad \blacksquare$$

**Def (Link coloring)** Given a ribbon Hopf alg $(A, m, \eta, \Delta, \varepsilon, S, R, \theta)$, and a link $\mathcal{L}$, a "colouring" of $\mathcal{L}$ by reps is an assignment

$$\{\mathcal{L}_1, \ldots, \mathcal{L}_m\} \longrightarrow \{V_1, \ldots, V_m\}$$

of reps $V_i$ of $A$ to link components $\mathcal{L}_i$ of $\mathcal{L}$.

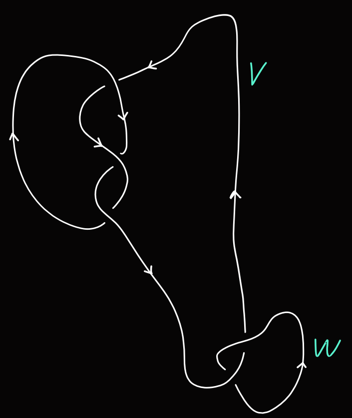
Denote a coloured link $\mathbb{V}(\mathcal{L})$.

**Worked Example (Intertwiner associated to coloured link)**

① Let $(A, m, \eta, \Delta, \varepsilon, S, R, \theta)$ be a ribbon Hopf alg w/ $S$ invertible; & $\mathbb{V}(\mathcal{L})$ be a coloured oriented framed link

② Embed $\mathcal{L}$ in $S^3 \cong \mathbb{R}^3 \cup \{\infty\}$; pick a plane in $S^3$ & isotopy $\mathcal{L}$ to have blackboard framing wrt the plane

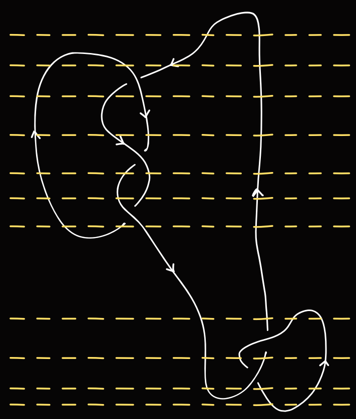
eg



③ Interpret $\mathbb{V}(\mathcal{L})$ as the string diagram encoding some $A$-linear morphism $\mathbb{F} \longrightarrow \mathbb{F}$; explicitly,

ⓐ isotopy $\mathbb{V}(\mathcal{L})$ st its projection decomposes into vertically-stacked slices, each containing exactly 1 of the following

eg

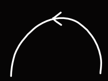


ⓑ associate $A$-linear homomorphisms as follows:

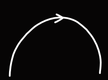
$c \equiv \tau \circ R$

$c^{-1}$

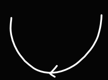
recall, with $R = \sum \alpha_j \otimes \beta_j$, $u = \sum S(\beta_j)\alpha_j$



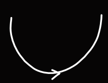
$1_{\mathbb{F}} \longmapsto \sum e^k \otimes u^{-1}\theta\, e_k$ $\qquad \mathbb{F} \longrightarrow V^* \otimes V$



$1_{\mathbb{F}} \longmapsto \sum e^k \otimes e_k$ $\qquad \mathbb{F} \longrightarrow V \otimes V^*$



$v \otimes \alpha \longmapsto \alpha(\theta^{-1} u v)$ $\qquad V \otimes V^* \longrightarrow \mathbb{F}$



$\alpha \otimes v \longmapsto \alpha(v)$ $\qquad V^* \otimes V \longrightarrow \mathbb{F}$

remark we can check from 1st principles that all of the above is $A$-linear, but it is messy

eg for $v \otimes \alpha \overset{\phi}{\longmapsto} \alpha(\theta^{-1} u v)$, we have

$$\phi(a \cdot (v \otimes \alpha)) = \phi(\Delta(a)(v \otimes \alpha)) \qquad \text{(def of } A\text{-axn on } V \otimes W)$$

$$= \phi\Big(\sum_{(a)} a' v \otimes a'' \alpha\Big) \qquad \text{(rearrangement)}$$

$$= \alpha\Big(\sum_{(a)} S(a'')\, \theta^{-1} u a' v\Big) \qquad \text{(defs of } \phi \text{ \& } A\text{-axn on } V^*)$$

$$= \alpha\Big(\sum_{(a)} S(a'')\, \theta^{-1} S^2(a') u v\Big) \qquad (S^2(-) = u(-)u^{-1})$$

$$= \alpha\Big(\sum_{(a)} S(a'') S^2(a')\, \theta^{-1} u v\Big) \qquad (\theta^{-1} \in \mathcal{Z}(A))$$

$$= \alpha\big(S(m(S\otimes id))\Delta(a)\,\theta^{-1}uv\big) \qquad \text{(rearrangement)}$$
$$= \alpha\big(S(\varepsilon(a)1)\,\theta^{-1}uv\big) \qquad \text{(def of } S)$$
$$= \varepsilon(a)\,\alpha(\theta^{-1}uv) \qquad \text{(linearity)}$$
$$= a\circ\phi(\alpha\otimes v) \qquad \text{(defs of } A\text{-axn on } \mathbb{F})$$

④ The result of ③ is some intertwiner $\psi: \mathbb{F}\to\mathbb{F}$ represented by $\mathbb{V}(\mathcal{L})$

**Def (Reshetikhin–Turaev invariant)** The Reshetikhin–Turaev invariant associated to an oriented framed link $\mathbb{V}(\mathcal{L})$, coloured by reps of some ribbon Hopf alg $(A, m, \eta, \Delta, \varepsilon, S, R, \theta)$ is the scalar $\langle\mathbb{V}(\mathcal{L})\rangle\in\mathbb{F}$ by which the morphism $\psi:\mathbb{F}\to\mathbb{F}$ encoded by $\mathbb{V}(\mathcal{L})$ as above multiplies inputs

ie. $\psi: z\longmapsto\langle\mathbb{V}(\mathcal{L})\rangle\, z$

**Thm** $\langle\mathbb{V}(\mathcal{L})\rangle$ is a bona fide link invariant

**Proof** (special case of coherence thm for ribbon cats) ∎

1st principles proof also found in Geka (2014), thm 8.2

**Thm (Identities for $\langle\mathbb{V}(\mathcal{L})\rangle$)** Denote $\mathbb{V}(L_i)=V_i$ for link cmp $L_i$; Then

① If $V_j = W_1\oplus W_2$, then $\langle\mathbb{V}(\mathcal{L})\rangle = \langle\mathbb{V}_1(\mathcal{L})\rangle+\langle\mathbb{V}_2(\mathcal{L})\rangle$, where

$$\mathbb{V}_i(L_k) = \begin{cases} V_k & k\neq j \\ W_i & k=j\end{cases}$$

② (cabling principle) If $V_j = W_1\otimes W_2$, then $\langle\mathbb{V}(\mathcal{L})\rangle = \langle\mathbb{V}'(\mathcal{L}')\rangle$, where
- $\mathcal{L}'$ is $\mathcal{L}$, but with $L_j \rightsquigarrow L_j^{\parallel 2}$
- $\mathbb{V}'$ is $\mathbb{V}$, but assigning $W_1$ & $W_2$ to the 2 cmps of $L_j^{\parallel 2}$

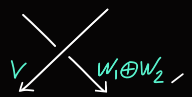
③ $\langle\mathbb{V}(\mathcal{L})\rangle = \langle\mathbb{V}'(\mathcal{L}')\rangle$, where
- $\mathcal{L}'$ is $\mathcal{L}$, but with some $L_j$ reversed
- $\mathbb{V}'$ is $\mathbb{V}$, but assigning $V_j^*$ to $L_j$

④ If $V_j$ is the trivial rep, then $\langle\mathbb{V}(\mathcal{L})\rangle = \langle\mathbb{V}'(\mathcal{L}')\rangle$, where
- $\mathcal{L}'$ is $\mathcal{L}$, but w/ $L_j$ deleted
- $\mathbb{V}'$ is $\mathbb{V}$, restricted to $\mathcal{L}'$

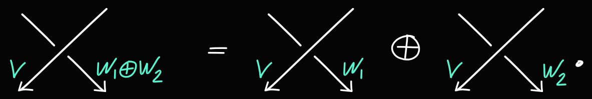
# Proof Sketch

① Note that $a\cdot(s\oplus t) = (as)\oplus(at)$, so the homomorphisms defined by

split across $\oplus$. For instance, for a crossing




we identify $V\otimes(W_1\oplus W_2) = V\otimes W_1 \oplus V\otimes W_2$, whence the endomorphism defined by $R$ acting on $V\otimes(W_1\oplus W_2)$ splits into endomorphisms on $V\otimes W_1$ and $V\otimes W_2$; the same is then true for $c$ & $c^{-1}$, so



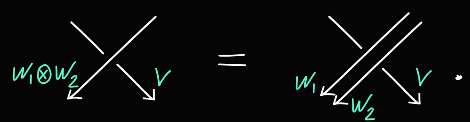

Similar reasoning gives splittings of the other homomorphisms.

② Consider first the homomorphism $c$ associated to

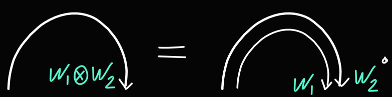
We compute

$$
\begin{aligned}
c_{W_1\otimes W_2,V} &= \tau_{W_1\otimes W_2,V}\, R_{W_1\otimes W_2,V} && \text{(def of } c\text{)}\\
&= \tau_{W_1\otimes W_2,V}\, (\Delta\otimes id)(R)_{(W_1\otimes W_2)\otimes V} && \text{(rearrangement)}\\
&= \tau_{W_1\otimes W_2,V}\, (R_{13}R_{23})_{W_1\otimes W_2\otimes V} && \text{(axiom of braided bialg)}\\
&= \sum_{j,k} \beta_j\beta_k\otimes\alpha_j\otimes\alpha_k && \text{(writing } R=\textstyle\sum \alpha_j\otimes\beta_j\text{)}\\
&= (c_{W_1,V}\otimes id)(id\otimes c_{W_2,V}) && \text{(def of } c\text{)}
\end{aligned}
$$

ie,




We can similarly easily check that




The case of $v\otimes\alpha \overset{\Phi}{\mapsto} \alpha(\theta^{-1}u\,v)$ requires a bit more work + relies on the other braided bialg property; we omit the details (see Gelca (2014), §8.1.4)

③ Reversing a strand + colouring w/ the dual encodes the same rep; the only "issues" are at maxima / minima, but our pivotal structure ensures consistency

④ Trivial rep $V^0 \equiv \mathbb{F}$; now $R: V\otimes\mathbb{F} \longrightarrow V\otimes\mathbb{F}$ is the map

$V \longrightarrow V: v \longmapsto \sum_j \varepsilon(\beta_j)\alpha_j v$, by def of $A$-axn on $\mathbb{F}$.

But $\sum_j \varepsilon(\beta_j)\alpha_j = m((\varepsilon\otimes id)(R)) = 1$ (thm from last time),

so $c: V\otimes\mathbb{F} \longrightarrow V\otimes\mathbb{F} \;\equiv id_V$.

Moreover, since $\varepsilon(\theta) = \varepsilon(u) = 1$, each of the other homomorphisms (cups + caps) also act as identity under identification $\mathbb{F}\otimes\mathbb{F} = \mathbb{F}$ ∎