Quaternions and 3D Rotations

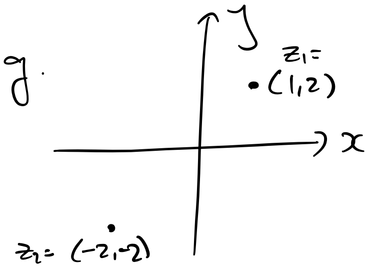
[Talk by Dario Trinchero on 25/4/'24]

↳ 4-dimensional number system

William Hamilton asked: How does one multiply and divide numbers in 3D space?

We know how to do this in 2D:

e.g.



$$
\begin{aligned}
z_1 z_2 &= (1+2i)(-2-2i) \\
&= -2-4i-2i-4(i\cdot i) \\
&= -2-6i+4 \\
&= 2-6i
\end{aligned}
$$

$$\frac{z_1}{z_2} = \frac{1+2i}{-2-2i} \cdot \frac{-2+2i}{-2+2i} = \frac{-2-4-4i+2i}{4+4} = \frac{-6-2i}{8} = -\frac{3}{4} + \frac{1}{4} i$$

An analogue for 3D does not exist!

↳ Proved by Frobenius in 1877: $\mathbb{R}^n$ can only be equipped with a product which supports division if one of the following are true:

- $n=1$: $\mathbb{R}$, the real numbers.
- $n=2$: $\mathbb{C}$, the complex numbers
- $n=4$: $\mathbb{H}$, the quaternions.

Quote by Hamilton: "And here dawned on me the notion that we must admit, in some sense, a fourth dimension of space for the purpose of calculating with triples..! An electric circuit seemed to close, and a spark flashed forth."

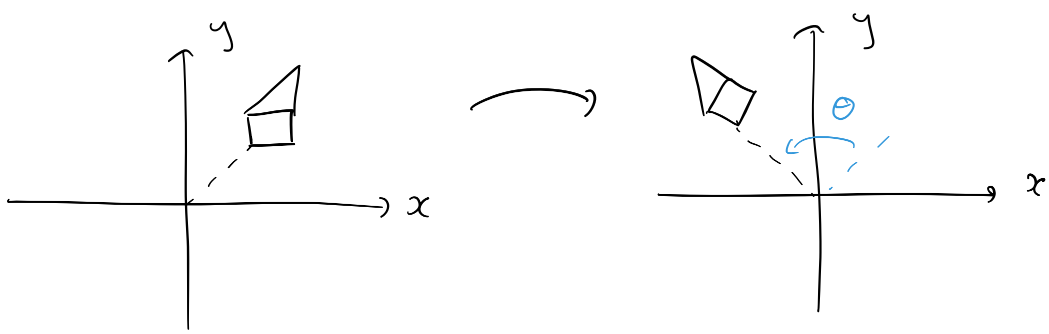
Useful in Physics: Bosons & fermions

e.g. electrons

when they rotate, we use quaternions to describe these rotations.

§1: Rotations in 2D.

Imagine rotating (counterclockwise) a shape in 2D by angle θ:



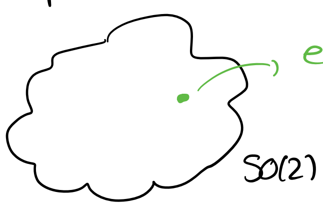
We have two questions:

① How do we calculate where each point is mapped?

② What does the space of rotations look like?

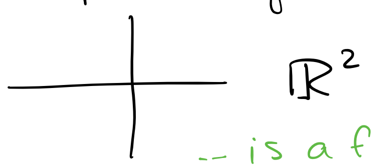
We have two spaces here:

1). space of rotations:



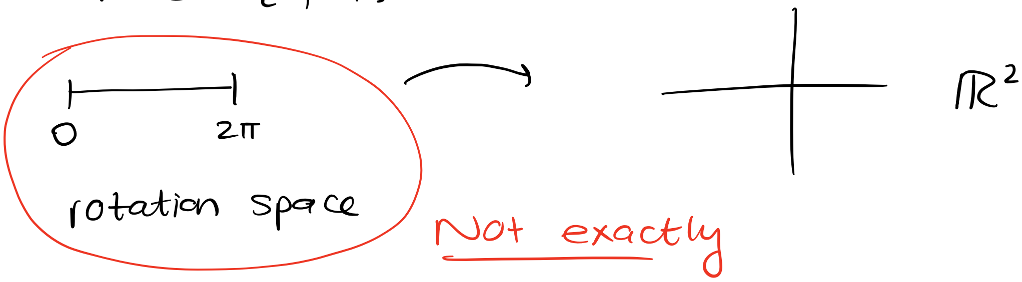
each point r here ...

2). space being rotated



... is a function $r: \mathbb{R}^2 \to \mathbb{R}^2$

Every 2D rotation is described by one number $\theta \in [0, 2\pi]$:

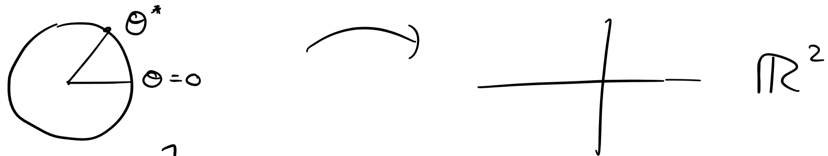


Not exactly

$r_0 = r_{2\pi}$

↳ so we glue the ends of the interval

We get :



$SO(2) = S^1$

But SO(2) also has an algebraic structure:

$r_\theta \circ r_\phi = r_{(\theta+\phi) \bmod 2\pi}$. $(\theta, \phi) \longmapsto (\theta+\phi) \bmod 2\pi$

on the level of maps $\mathbb{R}^2 \to \mathbb{R}^2$ $SO(2) \times SO(2) \longrightarrow SO(2)$

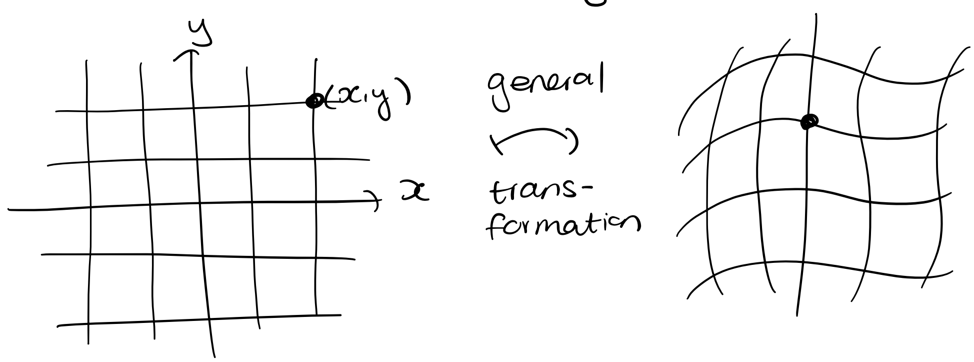
These operations make SO(2), as well as the collection $\{r_\theta : \mathbb{R}^2 \to \mathbb{R}^2 \mid \theta \in SO(2)\}$ into an algebraic structure, called a Group.

↳ A set with binary operation such that there is an identity

This answers ②. We now look at ①:

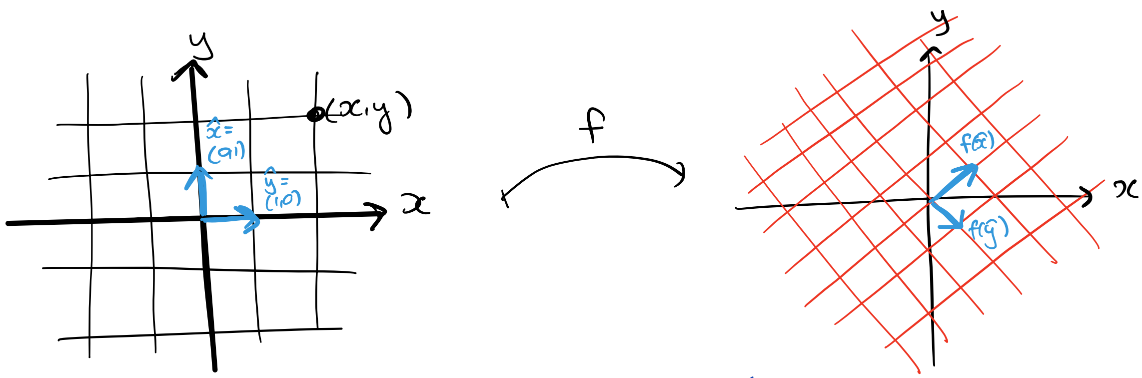
element, inverses to each element, and the binary operation is associative.

① Let's calculate $r_\theta(x,y)$:



Trans formations on 2D space can get messy, as seen above.

But Linear Trans formations keep our 'grid lines' parallel and equally-spaced, e.g.



(it is allowed for the lines to be closer together post transformation).

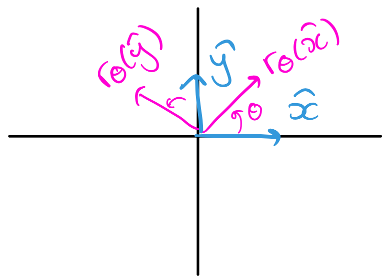
We only need to know where the trans formation sends $\hat{x}$ and $\hat{y}$ to be able to describe the entire trans formation.

This information can be encoded in a matrix:

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 2 \\ -1 \end{bmatrix} + y\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x-y \\ x \end{bmatrix}.$$

$$\boxed{\text{Rotations are linear transformations.}}$$

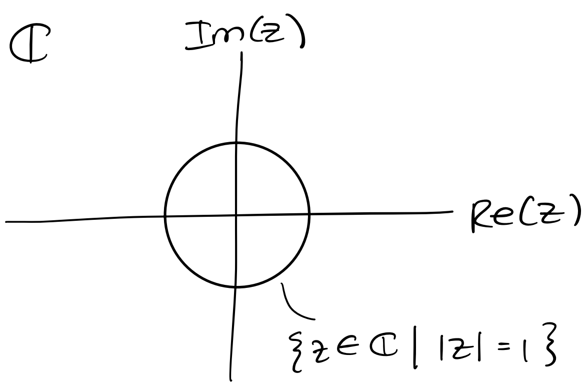
The matrix corresponding to rotation by θ is:



$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

But another way:

1. embed $SO(2)$ as a circle in the complex plane:



A point $z \in SO(2)$ has complex coordinates:

$$z_\theta = \cos\theta + i\sin\theta$$

So,

$$\begin{aligned} z_\theta z_\phi &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) \\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\theta) \\ &= \cos(\theta+\phi) + i\sin(\theta+\phi) = z_{\theta+\phi} \end{aligned}$$

② . embed $\mathbb{R}^2$ also in $\mathbb{C}$:

$$(x,y) \longmapsto x+iy.$$

Then the action of r_θ on (x,y) is ALSO just complex multiplication:

$$r_\theta(x,y) = (\cos\theta + i\sin\theta)\cdot(x+iy)$$
$$= (x\cos\theta - y\sin\theta) + i(x\sin\theta + y\cos\theta)$$

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + y\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$
$$= \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}.$$

The complex numbers miraculously capture rotations in 2D!

Coming up:

This situation is sligutly nicer than in 3D, but the following will hold:

We think of $\mathbb{R}^3$ as living in $\mathbb{H}$, as well as $SO(3)$, the space of 3D rotations.