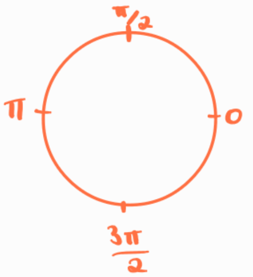
Quaternions

- ↳ 4D generalisation of complex numbers
- ↳ one of few generalisations that still let's you divide numbers
- ↳ application: rotations in 3D
  today's focus

1. Recap of space SO(2)

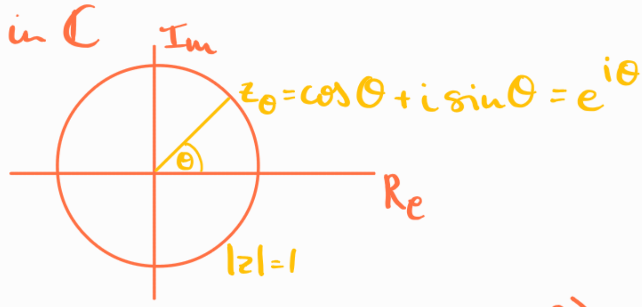
- has geometry of a circle



$$SO(2) = \left(\left\{ \begin{matrix} \text{rotations} \\ r_\theta : \mathbb{R}^2 \to \mathbb{R}^2 \end{matrix} \right\}, \circ, r_0 \right)$$

- can be realized as embedded in $\mathbb{C}$

$$SO(2) \cong \{ z \in \mathbb{C} \mid |z| = 1 \}$$



- group operations (composing rotations as maps $r_\theta : \mathbb{R}^2 \to \mathbb{R}^2$) corresponds to complex multiplication

$$r_\theta \circ r_\phi = r_{\theta+\phi} \qquad \mathbb{C}_{unit} = \left(\{ z \in \mathbb{C} \mid |z| = 1 \}, \times, 1 + 0i \right)$$

$$(\cos\theta + i\sin\theta) \cdot (\cos\phi + i\sin\phi) = \cos(\theta+\phi) + i\sin(\theta+\phi)$$

invertible map: $\rho : SO(2) \longrightarrow \mathbb{C}_{unit}$

$$r_\theta \longmapsto z_\theta = \cos\theta + i\sin\theta$$

s.t. $\rho(r_\theta) \times \rho(r_\phi) = \rho(r_\theta \circ r_\phi)$

– $\mathbb{R}^2$ can <u>also</u> be realised as embedded in $\mathbb{C}$

$$\mathbb{R}^2 \longrightarrow \mathbb{C}$$
$$(x, y) \longmapsto x + iy$$

– action of $z_0 \in \mathbb{C}$ on $x + iy$ is also just complex multiplication

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2. Quaternions

↳ a number $a+bi+cj+dk$ with $a,b,c,d \in \mathbb{R}$,
and i,j,k satisfying $i^2=j^2=k^2=ijk=-1$

Let $\mathbb{H} = \{a+bi+cj+dk \mid a,b,c,d \in \mathbb{R}\}$

Operations on $\mathbb{H}$

1) Addition:

$$(a+bi+cj+dk)+(e+fi+gj+hk) = (a+e)+(b+f)i+(c+g)j+(d+h)k$$

2) Multiplication: — direction dependent; not commutative

$$ijk=-1 \xRightarrow{i\times} i^2jk=-i=-jk \Rightarrow jk=i$$

$$ijk=-1 \xRightarrow{\times k} ijk^2=-k=-ij \Rightarrow ij=k$$

$$ij=k \xRightarrow{\times j} ij^2=kj=-i$$

↓×→	i	j	k
i	−1	k	−j
j	−k	−1	i
k	j	−i	−1

3) Division:

$$\frac{1}{a+bi+cj+dk} \times \frac{a-bi-cj-dk}{a-bi-cj-dk} = \frac{1}{a^2+b^2+c^2+d^2}(a-bi-cj-dk)$$

4) Norm:

$$|a+bi+cj+dk| = \sqrt{a^2+b^2+c^2+d^2}$$

Define

$$\mathbb{H}_{unit} = \{q \in \mathbb{H} \mid |q|=1\}$$

$$\mathbb{H}_{pure} = \{bi+cj+dk \mid b,c,d \in \mathbb{R}\}$$

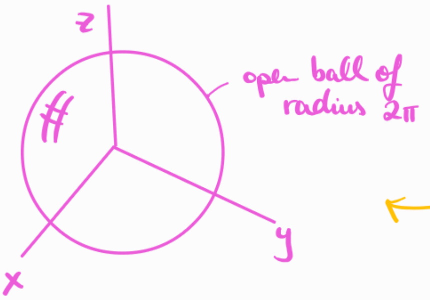
shorthand of $a+bi+cj+dk$ as $(a,\vec{x}) \in \mathbb{R}^4$ where $\vec{x}=(b,c,d) \in \mathbb{R}^3$

3. Rotation in 3D

Group of 3D rotations, $SO(3)$, is described by 2 parameters

1. <u>axis</u>: unit vector $\hat{n} \in \mathbb{R}^3$
2. <u>angle</u>: $\theta \in [0, 2\pi]$

Combine this into a single vector $\theta \cdot \hat{n}$



wrong visualisation

But $r(\hat{n}, \pi) = r(-\hat{n}, \pi)$ so really $SO(3)$ is a ball of radius π, with each pair of antipodal points "glued"

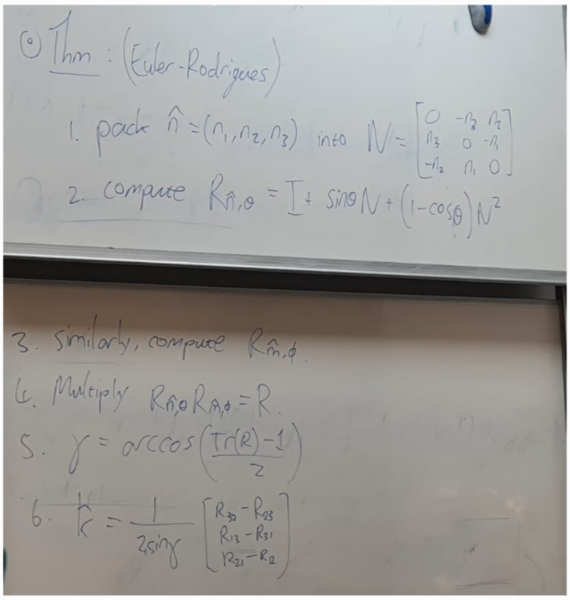
opposite points on sphere

Composing rotations

$$r(\hat{n}, \theta) \circ r(\hat{m}, \phi) = r(\hat{k}, \gamma)$$

how do we find these?

Thm. (Euler-Rodrigues)



4. Quaternions and rotations

1. embed $\mathbb{R}^3$ in $\mathbb{H}$ as $\mathbb{H}_{pure}$
2. think of rotations as elements of $\mathbb{H}_{unit}$ via the map

$$SO(3) \to \mathbb{H}_{unit}$$
$$(\hat{n},\theta) \longmapsto q_{\hat{n},\theta} = (\cos\tfrac{\theta}{2}, \sin\tfrac{\theta}{2}\hat{n})$$

3. action of $q_{\hat{n},\theta}$ on $\vec{x} \in \mathbb{H}_{pure}$ is given by

$$\vec{x} \longmapsto q_{\hat{n}\theta} \cdot \vec{x} \cdot q_{\hat{n}\theta}^{-1}$$

Example: $\theta = \frac{\pi}{3}$, $\hat{n} = \frac{1}{3}(1,2,2)$

$$q = \cos\tfrac{\pi}{6} + \sin\tfrac{\pi}{6}\left(\tfrac{1}{3}i + \tfrac{2}{3}j + \tfrac{2}{3}k\right)$$
$$= \tfrac{\sqrt{3}}{2} + \tfrac{1}{6}i + \tfrac{1}{3}j + \tfrac{1}{3}k$$
$$q^{-1} = \frac{1}{|q|^2}\bar{q} = 1 \cdot \bar{q} = \tfrac{\sqrt{3}}{2} - \tfrac{1}{6}i - \tfrac{1}{3}j - \tfrac{1}{3}k$$

So the rotation sends $xi+yj+zk$ to:

$$\left(\tfrac{\sqrt{3}}{2} + \tfrac{1}{6}i + \tfrac{1}{3}j + \tfrac{1}{3}k\right)(xi+yj+zk)\left(\tfrac{\sqrt{3}}{2} - \tfrac{1}{6}i - \tfrac{1}{3}j - \tfrac{1}{3}k\right)$$

Further references:

- eater.net/quaternions
- chapter 6 of Algebra and Geometry by A. Beardon