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Exploring Tensor Products

"A tensor is a thing that transforms like a tensor!"

Stellenbosch University
May 2021

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Where we are heading

We build to defining the tensor product as the unique space linearizing bilinear maps. We touch on more abstract settings, and the tensor-hom adjunction.

Definition (Vector space)

A complex VECTOR SPACE is a set X with element $0_X \in X$, and operations $+\colon X \times X \to X$ and $\cdot\colon \mathbb{C} \times X \to X$, satisfying, for all $x, y, z \in X$, $\lambda, \mu \in \mathbb{C}$,

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(the first four items together: $(X, +)$ is an Abelian group)

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Hereafter, take X, Y, Z to be complex vector spaces.

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Definition (Dual space)

The vector space X^* of all linear functionals $X \to \mathbb{C}$ is the DUAL SPACE of X. If X has basis $\mathcal{B} = (b_1, \ldots, b_n)$, then X^* has DUAL BASIS $(\beta^1, \ldots, \beta^n)$, where $\beta^j(b_i) := \delta_{ij}$.

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Definition (Quotient vector space)

For each $x \in S$, define the EQUIVALENCE CLASS $[x] := \{y \in S \colon y \sim x\}$, and the QUOTIENT SPACE $S/\sim := \{[x] \colon x \in S\}$. For vector space X, note that $X/\sim$ is also a vector space if we define $[x] + [y] := [x + y]$, $\lambda[x] := [\lambda x]$.

Component Definition

"A tensor is a thing that transforms like a tensor!"

Texts aimed at physicists[7] typically define:

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...... what??!

Examples of Tensors

Unpacking the mess

Let us look at 3 specific simple cases.

Example (Tensor of type $(1, 0)$)

1D array $[T_{\mathcal{B}}]^i$, which under $\mathcal{B} = (b_1, \ldots, b_n) \mapsto \left(b_i P^i_1, \ldots, b_i P^i_n\right) = \mathcal{C}$, transforms by $[T_{\mathcal{C}}]^{i'} = (P^{-1})^{i'}_i \, [T_{\mathcal{B}}]^i$

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- so the basis vectors transform $c_j = b_i \left(P_{\mathcal{B}\leftarrow\mathcal{C}}\right)_{ij}$ $\equiv b_i P^i_j$ from above
- but coordinates transform by the inverse matrix: $[v_{\mathcal{C}}] = P_{\mathcal{C}\leftarrow\mathcal{B}} \, [v_{\mathcal{B}}] = \left(P_{\mathcal{B}\leftarrow\mathcal{C}}\right)^{-1} [v_{\mathcal{B}}]$,

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- hence $[f_{\mathcal{C}}]_j = f(c_j) = f\left(b_i P^i_j\right) = f(b_i) P^i_j = [f_{\mathcal{B}}]_i \, P^i_j$

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- the components $[M_{\mathcal{B}}]_{ij}$ of linear operator M with respect to basis $\mathcal{B}$ are computed $[M_{\mathcal{B}}]_{ij} = [M(b_j)_{\mathcal{B}}]^i$
- applying previous examples, with linearity,

$$\begin{aligned}
[M_{\mathcal{C}}]_{ij} = [M(c_j)_{\mathcal{C}}]^i &= \left[\left(T(b_{j'}) P^{j'}_j\right)_{\mathcal{C}}\right] \\
&= \left(P^{-1}\right)^i_{i'} [M(b_{j'})_{\mathcal{B}}]^{i'} \, P^{j'}_j = \left(P^{-1}\right)^i_{i'} [M_{\mathcal{B}}]_{i'j'} \, P^{j'}_j
\end{aligned}$$

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Our examples suggest, informally,

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Theorem (Singular-value decomposition)

An $m \times n$ matrix A of rank $r \leq n$ may be written in the form

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T,$$

where the σ_i are called the "singular values" of A.

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Note (Formal linear combinations)

We commonly write $f \in F(S)$ as a formal linear combination

$$\sum_{x \in S} f(x)x$$

of elements of S. There are finitely-many non-0 coefficients, since f has finite support.

Quotient Space Definition

An admittedly only-slightly-more-elegant formulation

Definition 2 (Tensor product)

The TENSOR PRODUCT $X \otimes Y$ is the quotient space, $\mathsf{F}(X \times Y)/\sim$, of the free vector space by the equivalence relation $\sim$ generated by:

$$(x_1, y) + (x_2, y) \sim (x_1 + x_2, y), \qquad (x, y_1) + (x, y_2) \sim (x, y_1 + y_2)$$

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This formalizes the link between (co-/)contravariant indices & (co-)vectors.

Lemma (Basis for tensor product)

Given bases $\mathcal{B} = (b_1, \ldots, b_n), \mathcal{C} = (c_1, \ldots, c_m)$ for X and Y respectively, the vectors $b_i \otimes c_j$ form a basis for $X \otimes Y$.

Proof.

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Proof.

1. For linear indepdendence, note that $(b_i, c_j) \not\sim (b_{i'}, c_{j'})$ whenever $(i, j) \neq (i', j')$, or we would have two linearly-dependent vectors in either $\mathcal{B}$ or $\mathcal{C}$.

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Lemma (Basis for tensor product)

Given bases $\mathcal{B} = (b_1, \dots, b_n), \mathcal{C} = (c_1, \dots, c_m)$ for X and Y respectively, the vectors $b_i \otimes c_j$ form a basis for $X \otimes Y$.

Proof.

1. For linear independence, note that $(b_i, c_j) \not\sim (b_{i'}, c_{j'})$ whenever $(i, j) \neq (i', j')$, or we would have two linearly-dependent vectors in either $\mathcal{B}$ or $\mathcal{C}$.
2. The general element of $X \otimes Y$ has the form $\sum_i \lambda_i x_i \otimes y_i$, for $x_i \in X, y_i \in Y$. Expanding each x_i and y_i in the bases, and using linearity, we express this as a linear combination of vectors $b_i \otimes c_j$. ∎

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Definition (Components of tensor)

The COMPONENTS $[T_{\mathcal{B}}]^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ of tensor T of type (p, q) on X are its expansion coefficients wih respect to the induced basis on $X \otimes \cdots \otimes X \otimes X^* \otimes \cdots \otimes X^*$:

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Note (Transformation law)

It is clear that this satisfies the required transformation law under change-of-basis. Loosely, each b vector acquires a prefactor of P^{-1}, and each β a prefactor of P.

Quotient Space Definition

Recovering rank

Definition (Rank)

An ELEMENTARY TENSOR R (of type (p, q)) can be written as a single term $R = a_1 \otimes \cdots \otimes a_p \otimes \alpha_1 \otimes \cdots \otimes \alpha_q$, for non-0 $a_i \in X, \alpha_i \in X^*$. The RANK of tensor T is the minimum number of elementary tensors summing to T.

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Note (Rank)

This definition coincides with how the matrix rank (dimension of row-space) entered the singular value decomposition! A rank-r matrix was expressible thereby as a sum of r elementary terms $\sigma_i \mathbf{u}_i \mathbf{v}_i^T$.

Bilinear Maps

Definition (Bilinear maps & forms)

A BILINEAR MAP is a map $F \colon X \times Y \to Z$, which is linear in both arguments:

$$F(\lambda x_1 + \mu x_2, y) = \lambda F(x_1, y) + \mu F(x_2, y)$$
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Notation (Bilinear maps & forms)

Write $\mathrm{B}(X \times Y, Z)$ for the vector space of bilinear maps into Z, and $\mathrm{B}(X \times Y)$ for that of bilinear forms.

Bilinear Form Definitions

Two ways!

Just as linear functionals are type-$(0, 1)$ tensors, we can think of bilinear forms on $X \times X$ as type-$(0, 2)$ tensors on X. This motivates the definition in Jeevanjee[1]:

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The definition in Ryan[6] looks subtly different:

Definition 5 (Tensor product)

The TENSOR PRODUCT $X \otimes Y$ is the subspace of $\mathrm{B}(X \times Y)^*$ spanned by functionals $x \otimes y$, defined

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Note (Linking via quotient space)

Both definitions are manifestly spanned by pairs $x \otimes y$, and both produce pairs bilinear in x & y. We can thus exhibit obvious isomorphisms from each of them to $\mathsf{F}(X \times Y)/\sim$, proving all 3 definitions isomorphic.

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Alternatively, we can use the linearizing properties we showed. Use subsript R for Ryan and J for Jeevanjee.

Note (Linking via linearization)

The map $X \times Y \to \mathsf{B}(X^* \times Y^*)\colon (x, y) \mapsto x \otimes_J y$, with $(x \otimes_J y)(f, g) := f(x)g(y)$, is manifestly bilinear. Hence there is unique linear map $l\colon X \otimes_R Y \to \mathsf{B}(X^* \times Y^*)$, easily shown to be injective. So $X \otimes_R Y$ embeds in $\mathsf{B}(X^* \times Y^*)$, & is isomorphic to the subspace spanned by $x \otimes_J y$.

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Unifying all prior definitions

More abstractly,

Definition 6 (Tensor product)

The TENSOR PRODUCT $X \otimes Y$ is the vector space and associated bilinear map $\phi\colon X \times Y \to X \otimes Y$ satisfying the following universal property:

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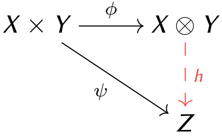

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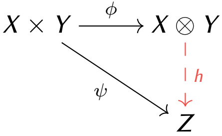

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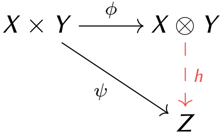

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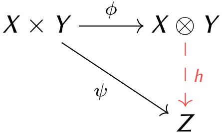

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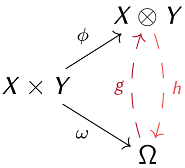

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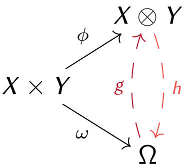

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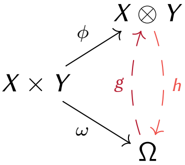

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4. Then $g \circ h \circ \phi = \phi$, so $g \circ h = 1$ on $\phi(X \times Y)$, and hence on its span, $X \otimes Y$. Similarly, $h \circ g = 1$ on Ω, so $h^{-1} = g$. ■

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Briefly wandering out of scope

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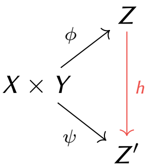

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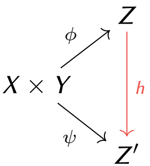

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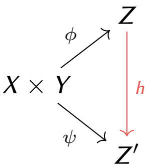

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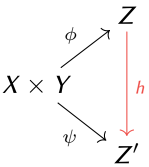

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- harkens to connection between universal arrows & universal objects in some comma category

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We can think of a bilinear map $\phi \in \mathrm{B}(X \times Y, Z)$ instead as a linear map $\phi' \colon X \to \mathrm{Hom}(Y, Z) \colon x \mapsto (y \mapsto \phi(x, y))$. In this way, we identify $\mathrm{B}(X \times Y, Z)$ with $\mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$.

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We can now succinctly formulate the linearizing property of $X \otimes Y$:

Theorem 7 (Tensor-hom adjunction)

$\mathrm{Hom}(X, \mathrm{Hom}(Y, Z)) \cong \mathrm{Hom}(X \otimes Y, Z)$.
We say the tensor product $- \otimes Y$ *is* LEFT-ADJOINT *to the internal hom functor* $\mathrm{Hom}(Y, -)$.

Example (Cauchy stress tensor)

A common first-encounter with tensors is the CAUCHY STRESS TENSOR in classical mechanics. For direction vector $\mathbf{n} \in \mathbb{R}^3$, the 9-component stress tensor σ_{ij} gives the traction vector: $T_j^{(n)} = \sigma_{ij} n_i$.

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Example (Tensor product states)

In quantum mechanics, the set of states of a system is given by a Hilbert space. The two systems taken together have Hilbert space the tensor product of the individual systems' Hilbert spaces.

Entanglement arises precisely because not all tensors in this product are elementary.

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What we have shown

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We saw some examples of tensors, including from physics.

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