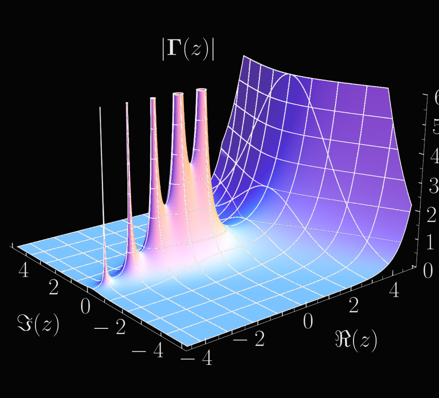
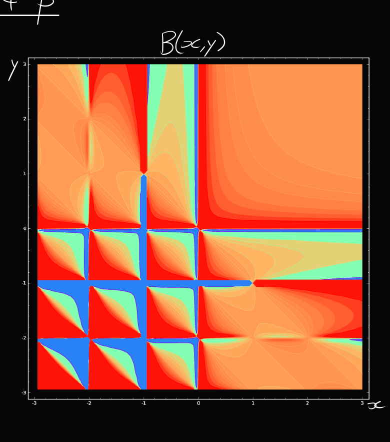
Mathematics 244 Tutorial 4

Analysis

§I – The Rich Mathematics of Γ & β





Recall, $\boxed{\Gamma:(0,\infty)\longrightarrow\mathbb{R}: z\longmapsto\int_0^\infty e^{-x}x^{z-1}\,dx}$ — DEF: Euler Integral of 2nd Kind

& $\Gamma(z) = 2\int_0^\infty e^{-x^2}x^{2z-1}\,dx = \int_0^1\left(\ln\frac{1}{x}\right)^{z-1}dx$
$= a^z\int_0^\infty e^{-ax}x^{z-1}\,dx \qquad a>0$
⊛ $= (z-1)\Gamma(z-1) \qquad z\in(1,\infty)$
⊛ $= \pi^{-\frac12}2^{z-1}\,\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right)$
$= (z-1)! \qquad z-1\in\mathbb{N}$

THM: Alternate Forms — integrals — recurrence relations — factorial

⊛ These allow extension of Γ to $\mathbb{R}\setminus\{0,-1,-2,\dots\}$ (!)

Our def. actually works on $\{z\in\mathbb{C}\mid\mathrm{Re}(z)>0\}$, so hereby we extend to $\mathbb{C}\setminus\{0,-1,-2,\dots\}$ (!!)

Even better, this extension is
① $\mathbb{C}$-differentiable ("holomorphic") everywhere
② unique in satisfying ① (!!!)

Neat Results

① $1 = 0! = \Gamma(1) = \pi^{-\frac12}2^{1-1}\,\Gamma\left(\frac12\right)\Gamma\left(\frac{1+1}{2}\right) = \pi^{-\frac12}\Gamma\left(\frac12\right)$
$\Longrightarrow \Gamma\left(\frac12\right) = \sqrt{\pi}$ "$\frac12! = \sqrt{\pi}$", loosely

② $\Gamma(z)\Gamma(1-z) = \dfrac{\pi}{\sin(\pi z)} \Longrightarrow \sin(\pi z) = \pi z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$

③ $\zeta(z) = \dfrac{1}{\Gamma(z)}\displaystyle\int_0^\infty\frac{x^{z-1}}{e^x-1}\,dx$ Famous Riemann ζ-function!

This is too advanced for 244

Practical Uses

Physics

Dimensional Regularization

The idea of dimensional regularization is very simple to state: Compute the Feynman diagram as an analytic function of the dimensionality of space-time, d. For sufficiently small d, any loop-momentum integral will converge and therefore the Ward identity can be proved. The final expression for any observable quantity should have a well-defined limit as $d\to4$.

Let us do a practice calculation to see how this technique works. We consider spacetime to have one time dimension and $(d-1)$ space dimensions. Then we can Wick-rotate Feynman integrals as before, to give integrals over a d-dimensional Euclidean space. A typical example is

$$\int\frac{d^d\ell_E}{(2\pi)^d}\frac{1}{(\ell_E^2+\Delta)^2} = \int\frac{d\Omega_d}{(2\pi)^d}\cdot\int_0^\infty d\ell_E\,\frac{\ell_E^{d-1}}{(\ell_E^2+\Delta)^2}. \tag{7.80}$$

The first factor in (7.80) contains the area of a unit sphere in d dimensions. To compute it, use the following trick:

$$(\sqrt{\pi})^d = \left(\int dx\,e^{-x^2}\right)^d = \int d^dx\,\exp\left(-\sum_{i=1}^d x_i^2\right)$$
$$= \int d\Omega_d\int_0^\infty dx\,x^{d-1}e^{-x^2} = \left(\int d\Omega_d\right)\cdot\frac12\int_0^\infty d(x^2)\,(x^2)^{\frac d2-1}e^{-(x^2)}$$
$$= \left(\int d\Omega_d\right)\cdot\frac12\Gamma(d/2).$$

Peskin & Schroeder – Intro to QFT

Statistics

DEFINITION 4.9 A random variable Y is said to have a *gamma distribution with parameters* $\alpha>0$ and $\beta>0$ if and only if the density function of Y is

$$f(y) = \begin{cases}\dfrac{y^{\alpha-1}e^{-y/\beta}}{\beta^\alpha\Gamma(\alpha)}, & 0\le y<\infty,\\ 0, & \text{elsewhere,}\end{cases}$$

where

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}\,dy.$$

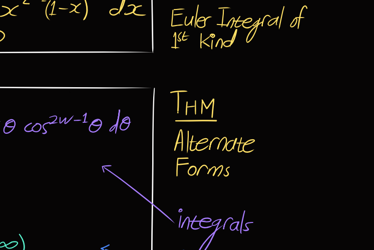


FIGURE 4.16 Gamma density functions, $\beta=1$

Wackerly, et al – Mathematical Statistics

Number Theory

Lemma 3 *If $c>0$ and $u>0$, then for every integer $k\ge1$ we have*

$$\frac{1}{2\pi i}\int_{c-\infty i}^{c+\infty i}\frac{u^{-z}}{z(z+1)\cdots(z+k)}\,dz = \begin{cases}\dfrac{1}{k!}(1-u)^k & \text{if } 0<u\le1,\\ 0 & \text{if } u>1,\end{cases}$$

the integral being absolutely convergent.

PROOF. First we note that the integrand is equal to $u^{-z}\Gamma(z)/\Gamma(z+k+1)$. This follows by repeated use of the functional equation $\Gamma(z+1)=z\Gamma(z)$. To prove the lemma we apply Cauchy's residue theorem to the integral

$$\frac{1}{2\pi i}\int_{C(R)}\frac{u^{-z}\Gamma(z)}{\Gamma(z+k+1)}\,dz,$$

where $C(R)$ is the contour shown in Figure 13.1(a) if $0<u\le1$, and that in Figure 13.1(b) if $u>1$. The radius R of the circle is greater than $2k+c$ so all the poles at $z=0,-1,\dots,-k$ lie inside the circle.

Apostol – Intro to Analytic Number Theory

…and many more!

Recall, $\boxed{\beta:(0,\infty)^2\longrightarrow\mathbb{R}:(z,w)\longmapsto\int_0^1 x^{z-1}(1-x)^{w-1}\,dx}$ — DEF: Euler Integral of 1st Kind

& $\beta(z,w) = \int_0^\infty\frac{x^{z-1}}{(1+x)^{z+w}}\,dx = 2\int_0^{\pi/2}\sin^{2z-1}\theta\cos^{2w-1}\theta\,d\theta$
$= \beta(w,z)$
$= \frac{z-1}{z+w-1}\beta(z-1,w) \qquad z\in(1,\infty)$
⊛ $= \dfrac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$

THM: Alternate Forms — integrals — functional equations — gamma

⊛ This links β to all of the theory above!

(1.a) *Show that $\int_0^\infty e^{-x^2}\,dx$ is convergent.*

$x\le x^2$ for all $x\in[1,\infty)$
$\Longrightarrow e^{-x^2}\le e^{-x}$ "
and $\int_1^\infty e^{-x}\,dx<\infty$ p-int
so $\int_0^\infty e^{-x^2}dx = \int_0^1 e^{-x^2}dx + \int_1^\infty e^{-x^2}dx<\infty$ cmp test

(1.b) *Now use Theorem 2(5) to calculate the integral in (a) above.*

$\left(2\int_0^\infty e^{-x^2}x^{2z-1}\,dx\right)$ $\quad\int_0^\infty e^{-x^2}dx = \frac12\,2\int_0^\infty e^{-x^2}x^{2(\frac12)-1}dx$
$= \frac12\Gamma\left(\frac12\right) = \frac{\sqrt{\pi}}{2}$

(2.a) $\int_0^\infty u^6e^{-3u}\,du = \Gamma(7)/3^7 = 6!/3^7$

(2.b) $\int_0^\infty\sqrt{x}\,6^{-3x}\,dx = \int_0^\infty x^{3/2-1}e^{-3\ln6\,x}\,dx$
$= \dfrac{\Gamma(3/2)}{(3\ln6)^{3/2}} = \dfrac{\frac12\Gamma(\frac12)}{(3\ln6)^{3/2}} = \dfrac{\sqrt{\pi}}{2(3\ln6)^{3/2}}$

(2.c) $\int_0^\infty x^3e^{-x}\,dx$ — parts would be painful
$= \Gamma(4) = 3! = 6$

(2.d) $\int_0^a y^4\sqrt{a^2-y^2}\,dy \quad (y^2=a^2x)\quad 2y\,dy = a^2\,dx$
$= \int_0^1 a^2x\,(a^2x)^{1/2}\sqrt{a^2-a^2x}\;\frac{1}{2}a^2\,dx$
$= \frac12\int_0^1 a^6x^{3/2}(1-x)^{1/2}\,dx = \frac{a^6}{2}\beta\left(\frac52,\frac32\right) = \dfrac{a^6\,\Gamma(\frac52)\Gamma(\frac32)}{2\,\Gamma(4)}$
$= \dfrac{a^6\,\cancel{3}\,\Gamma(\frac32)^2}{2^2\cdot\cancel{3}\cdot2} = \dfrac{a^6\pi}{2^5}$

(2.e) $\int_0^1\sqrt{\frac{1-x}{x}}\,dx = \int_0^1(1-x)^{1/2}x^{-1/2}\,dx$
$= \beta\left(\frac12,\frac32\right) = \dfrac{\sqrt{\pi}\,\frac12\sqrt{\pi}}{1!} = \dfrac{\pi}{2}$

(2.f) $\int_0^2\frac{x^2}{\sqrt{2-x}}\,dx \quad (x=2v)\quad dx = 2\,dv$
$= \int_0^1(2v)^2(2-2v)^{-1/2}\,2\,dv$
$= 2^{5/2}\beta\left(3,\frac12\right) = 2^{5/2}\dfrac{\Gamma(3)\sqrt{\pi}}{\Gamma(7/2)} = \dfrac{\cancel{\sqrt{\pi}}\,2^{13/2}}{15\cancel{\sqrt{\pi}}}$
$\left(\frac52\,\frac32\,\frac12\sqrt{\pi}\right.$

(2.g) $\int_0^1 x^4(1-x)^3\,dx = \beta(5,4) = \dfrac{4!\,3!}{8!}$

(2.h) $\int_{-1}^1\frac{1-x}{\sqrt{1+x}}\,dx \quad (2u=1+x)\quad 2\,du = dx$
$= \int_0^1(-2u+2)(2u)^{-1/2}\,du$
$= 2^{3/2}\beta\left(\frac12,2\right) = 2^{3/2}\dfrac{\sqrt{\pi}}{\Gamma(5/2)}$
$= 2^{3/2}\dfrac{2^2}{3} = \dfrac{2^{7/2}}{3}$

(2.i) $\int_{-\infty}^0 e^{2x}\,dx = \lim_{t\to-\infty}\int_t^0 e^{2x}\,dx \qquad u=-x,\ du=-dx$
$= \lim_{t\to-\infty}\int_{-t}^0 e^{-2u}(-du) = \lim_{-t\to\infty}\int_0^{-t}e^{-2u}\,du$
$= \int_0^\infty e^{-2u}\,du = \int_0^\infty e^{-2u}u^{1-1}\,du = \Gamma(1)/2 = \frac12$

(2.j) $\int_0^3\frac{1}{\sqrt{3x-x^2}}\,dx \quad (x=3u)\quad dx = 3\,du$
$= \int_0^1(9u-9u^2)^{-1/2}\,3\,du = \int_0^1 u^{-1/2}(1-u)^{-1/2}\,du$
$= \beta\left(\frac12,\frac12\right) = \pi$

(2.k) $\int_0^{\frac\pi2}\cos^5\theta\cdot\sin^2\theta\,d\theta = \frac12\beta\left(\frac32,3\right)$
$= \frac12\,\frac{1}{\cancel{2}}\,\dfrac{\sqrt{\pi}\,\cancel{2}}{\Gamma(9/2)} = \frac12\,\dfrac{\cancel{\sqrt{\pi}}\,2^4}{7\cdot5\cdot3\,\cancel{\sqrt{\pi}}}$

(2.l) $\int_0^{\frac\pi2}\cos^6\theta\,d\theta = \frac12\beta\left(\frac12,\frac72\right)$
$= \frac12\,\dfrac{\sqrt{\pi}\,\Gamma(7/2)}{6}$

(3.a) $\int_0^\infty 5^{-3x^2}\,dx \qquad u=x^2,\ du=2x\,dx \Longrightarrow dx = \frac12u^{-1/2}\,du$
$= \int_0^\infty e^{-(3\ln5)u}\,\frac{u^{-1/2}}{2}\,du$
$= \dfrac{\Gamma(\frac12)}{2(3\ln5)^{1/2}}$ — $\sqrt{\pi}$

(3.b) $\int_0^{\frac\pi2}\sin^4\theta\cos^5\theta\,d\theta = \frac12\beta\left(\frac52,3\right)$
$= \frac{1}{\cancel{2}}\,\dfrac{\cancel{2}\,\Gamma(5/2)}{\Gamma(11/2)} = \dfrac{\cancel{\Gamma(5/2)}}{\frac92\,\frac72\,\frac52\,\cancel{\Gamma(5/2)}}$
$= \dfrac{8}{9\cdot7\cdot5}$

(3.c) $\int_0^\infty\sqrt{x}\,3^{-2x}\,dx = \Gamma(3/2)/(2\ln3)^{3/2}$
$= \dfrac{\sqrt{\pi}}{2(2\ln3)^{3/2}}$

(4) *Prove*

Theorem 1. *The integral $I=\int_0^\infty\underbrace{e^{-x}x^{n-1}}_{f(x)}\,dx$ is convergent if n is a positive real number, and divergent if $n\le0$.*

$I = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx$

Case $n>0$

Take $g(x) = \frac{1}{x^2}$, whence $\int_1^\infty g(x)\,dx<\infty$ p-int
and $\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x^{n+1}}{e^x} \overset{LH}{=} \lim_{x\to\infty}\frac{(n+1)x^{n-1}}{e^x}$
$\underbrace{\overset{LH}{=}\cdots\overset{LH}{=}}_{\text{induction on }\lceil n\rceil}0$
$\Longrightarrow\int_1^\infty f(x)\,dx<\infty$ quotient test
If $n\ge1$, then $\int_0^1 f(x)\,dx<\infty$ clearly, so consider $0<n<1$:

$e^{-x}x^{n-1}\le x^{n-1}$, and $\int_0^1\frac{1}{x^{1-n}}\,dx<\infty$ p-int
$\Longrightarrow\int_0^1 f(x)\,dx<\infty$

Case $n\le0$

Take $g(x) = \frac{1}{x^{1-n}}$; $\int_0^1 g(x)\not<\infty$ p-int and
$\lim_{x\to0^+}\frac{f(x)}{g(x)} = \lim_{x\to0^+}e^{-x} = 1 \Longrightarrow\int_0^1 f(x)\,dx\not<\infty$ quotient test

(5) *Prove*

$\Gamma(n+1)=n!$ *for each $n\in\mathbb{N}$.*

Base case: $\Gamma(1+1) = \Gamma(2) = 1\,\Gamma(1) = \int_0^\infty e^{-x}\cancel{x^{1-1}}\,dx = 1$

Inductive step: $\Gamma((n+1)+1) = \Gamma(n+2) = (n+1)\Gamma(n+1)$
$\overset{I.H.}{=}(n+1)n! = (n+1)!$

(6) *Prove*

$\Gamma(n)=\int_0^1\left(\ln\frac1y\right)^{n-1}dy$ *for each $n\in(0,\infty)$.* — compare $(-\ln y)^{n-1}$ with x^{n-1}

$\Gamma(n) = \lim_{s\to0^+}\int_s^1 e^{-x}x^{n-1}\,dx + \lim_{t\to\infty}\int_1^t e^{-x}x^{n-1}\,dx$

$x=-\ln y\iff y=e^{-x}$
$dy = -e^{-x}\,dx$

$= \lim_{s\to0^+}\int_{e^{-s}}^{e^{-1}}-(-\ln y)^{n-1}\,dy + \lim_{t\to\infty}\int_{e^{-1}}^{e^{-t}}-(-\ln y)^{n-1}\,dy$
$= \int_{e^{-1}}^1\left(\ln\frac1y\right)^{n-1}dy + \int_0^{e^{-1}}\left(\ln\frac1y\right)^{n-1}dy$
$= \int_0^1\left(\ln\frac1y\right)^{n-1}dy$

(6) *Suppose $\{a_n\}$ and $\{b_n\}$ are convergent sequences with limits a and b respectively. Prove $\lim_{n\to\infty}(a_n+b_n)=a+b$*

$a_n\to a\iff\forall\varepsilon>0:\exists N_1\in\mathbb{N}: n\ge N_1\Longrightarrow|a_n-a|<\varepsilon/2$
$b_n\to b\iff\forall\varepsilon>0:\exists N_2\in\mathbb{N}: n\ge N_2\Longrightarrow|b_n-b|<\varepsilon/2$
Want $\forall\varepsilon>0:\exists N\in\mathbb{N}: n\ge N\Longrightarrow|(a_n+b_n)-(a+b)|$
$= |(a_n-a)+(b_n-b)|<\varepsilon$
$\le|a_n-a|+|b_n-b|$

Take $\varepsilon>0$. Since $a_n\to a$, $\exists N_1\in\mathbb{N}: n\ge N_1\Longrightarrow|a_n-a|<\varepsilon/2$. Since $b_n\to b$, $\exists N_2\in\mathbb{N}: n\ge N_2\Longrightarrow|b_n-b|<\varepsilon/2$. Take $N=\max\{N_1,N_2\}$. Then if $n\ge N$, (then $n\ge N_1$ and $n\ge N_2$, so) $|(a_n+b_n)-(a+b)| = |(a_n-a)+(b_n-b)|\le|a_n-a|+|b_n-b|<\varepsilon/2+\varepsilon/2<\varepsilon$.
∎

(7) *Suppose that $\{c_n\}$ is a convergent sequence in the domain $\mathcal{D}_f$ of f with limit $c\in\mathcal{D}_f$ and f is continuous at c. Prove that $\lim_{n\to\infty}f(c_n)=f(c)$.*

$c_n\to c\iff\forall\varepsilon>0:\exists N_1\in\mathbb{N}:\boxed{n\ge N_1}\Longrightarrow\boxed{|c_n-c|<\varepsilon}$
f cts at $c\iff\forall\varepsilon>0:\exists\delta>0:\boxed{|x-c|<\delta}\Longrightarrow\boxed{|f(x)-f(c)|<\varepsilon}$
Want $\forall\varepsilon>0:\exists N\in\mathbb{N}:\boxed{n\ge N}\Longrightarrow\boxed{|f(c_n)-f(c)|}<\varepsilon$

Take $\varepsilon>0$. Since f cts at c, $\exists\delta>0: |x-c|<\delta\Longrightarrow|f(x)-f(c)|<\varepsilon$. Since $c_n\to c$, $\exists N_1\in\mathbb{N}: n\ge N_1\Longrightarrow|c_n-c|<\delta$. Hence if $n\ge N_1$, then $|c_n-c|<\delta$, so $|f(c_n)-f(c)|<\varepsilon$. ∎

(8.a) $1,-\frac35,\frac59,-\frac{7}{13},\dots$
$a_n = (-1)^{n-1}\left(\frac{2n-1}{4n-3}\right)$
$|a_n| = \dfrac{2-1/n}{4-3/n}\longrightarrow\frac12\ne0$ as $n\to\infty$
so $\lim_{n\to\infty}a_n$ does not exist

(8.b) $a_1=2,\ a_{n+1}=\frac45a_n$
$\Longrightarrow\lim_{n\to\infty}a_{n+1} = \frac45\underbrace{\left(\lim_{n\to\infty}a_n\right)}_{\equiv L,\text{ say}}$ if lim exist
$\Longrightarrow L=\frac45L$
$\Longrightarrow L=0$

$a_n = \left(\frac45\right)^{n-1}2\longrightarrow0$ as $n\to\infty$

(9.a) $\frac{n!}{2^n} = \dfrac{n(n-1)(n-2)\cdots2\cdot1}{\underbrace{2\cdot2\cdot2\cdots2\cdot2}_{n\ 2\text{'s}}}\ge\frac n2\,\frac12$ for $n\ge4$
and $\frac n4\to\infty$ as $n\to\infty$, so $\frac{n!}{2^n}\to\infty$ likewise

(9.b) $\dfrac{\sqrt{n+1}-\sqrt{n}}{1}\times\dfrac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \dfrac{\cancel{n}+1-\cancel{n}}{\sqrt{n+1}+\sqrt{n}}\longrightarrow0$
as $n\to\infty$

(9.c) $n\cos n\pi = (-1)^nn$, which diverges, since $n\not\to0$
as $n\to\infty$

(9.d) $n\left(2^{\frac1n}-1\right)$ "$\longrightarrow0\cdot\infty$"
$\lim_{n\to\infty}\dfrac{2^{1/n}-1}{1/n}\overset{LH}{=}\lim_{n\to\infty}\dfrac{2^{1/n}(\ln2)\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{\left(-\frac{1}{n^2}\right)}} = \ln(2)$

(9.e) $2\ln3n-\ln(n^2+1) = \ln\left(\dfrac{9n^2}{n^2+1}\right)$
$= \ln\left(\dfrac{9}{1+1/n^2}\right)$, — $9\in(0,\infty)$
and $\ln$ is cts on $(0,\infty)$, so
$\lim_{n\to\infty}\ln\left(\dfrac{9}{1+1/n^2}\right) = \ln\left(\lim_{n\to\infty}\dfrac{9}{1+1/n^2}\right) = \ln(9) = 2\ln3$