Mathematics 214 Tutorial 5

Part I: Linear Algebra

Definition

A list of vectors $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ in vector space V

- **spans** V if every vector $\mathbf{v} \in V$ is a linear combination of vectors from $\mathcal{B}$
- is **linearly independent** if $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots k_n\mathbf{v}_n = \mathbf{0}$ has only trivial solution $k_1 = k_2 = \cdots = k_n = 0$
- is a **basis** for V if it is linearly independent and spans V

Proposition 2.2.10 The Steinitz Exchange Lemma. *Suppose $\mathcal{L} = \{\mathbf{l}_1, \mathbf{l}_2, \ldots, \mathbf{l}_m\}$ is a linearly independent list of vectors in a vector space V, and that $\mathcal{S} = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n\}$ spans V. Then $m \leq n$.*

if $\mathcal{B}$ is a basis, then $\dim V := |\mathcal{B}|$

Theorem 2.3.2 Invariance of dimension. *If $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_m\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n\}$ are bases of a vector space V, then $m = n$.*

(5) Suppose that V is an n-dimensional vector space. Use The Steinitz Replacement Lemma to show that: Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ be any basis of V.

(5a) any subset of V with less than n elements does not span V

Suppose $\mathcal{C} = \{u_1, \ldots, u_m\}$, where $m < n$, spans V. But $\mathcal{B}$ is linearly independent (as it is a basis), so (2.2.10) gives $m \geq n$ # $\square$

(5.b) any subset of V with more than n elements is not linearly independent

Suppose $\mathcal{C} = \{u_1, \ldots, u_m\}$ where $m > n$ is lin-indep. But $\mathcal{B}$ spans V (as it is a basis), so (2.2.10) gives $n \geq m$ # $\square$

(1) Is $\{\underbrace{1 + x + x^2}_{p_1}, \underbrace{1 + 2x + 3x^2}_{p_2}, \underbrace{1 + 3x + 5x^2}_{p_3}\}$ a basis for $Poly_2$?

Check linear independence: Suppose $c_1p_1 + c_2p_2 + c_3p_3 = 0$, whence

equate coeff's
$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\vec{c}} = \vec{0}.$$
By fundamental theorem of invertible matrices, we have unique solution $\vec{c} = \vec{0}$ iff $\det A \neq 0$.

$$\det A = 1\begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} + 1\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1 - 2 + 1 = 0.$$

Hence $\sum c_n p_n = 0 \not\Rightarrow \forall n: c_n = 0$, so $\{p_1, p_2, p_3\}$ is linearly dependent; (ie. not a basis).

(2) Determine which of the following sets is a basis for $V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 - x_2 + x_3 - x_4 = 0\}$: ← we expect $\dim V = 3$

(2a) $S_1 = \{(1, 1, 1, 1), (0, 1, 1, 0)\}$

$$V = \{(\lambda, \mu, \eta, \lambda - \mu + \eta) \mid \lambda, \mu, \eta \in \mathbb{R}\}$$
$$= \{\lambda \underbrace{(1, 0, 0, 1)}_{v_1} + \mu \underbrace{(0, 1, 0, -1)}_{v_2} + \eta \underbrace{(0, 0, 1, 1)}_{v_3} \mid \lambda, \mu, \eta \in \mathbb{R}\}$$
$$= \operatorname{span}\{v_1, v_2, v_3\}$$

Suppose $c_1v_1 + c_2v_2 + c_3v_3 = \vec{0}$, whence $c_1 = 0$, $c_2 = 0$, $c_3 = 0$, so $\{v_1, v_2, v_3\}$ are a basis for V! Hence by (2.3.2), S_1 cannot be a basis.

(2.b) $S_2 = \{(1, 1, 1, 1), (0, 1, 1, 0), (1, 0, 0, 0)\}$ ← $\notin V$

S_2 is not a basis since $(1, 0, 0, 0) \notin V$.

(2.c) $S_3 = \{\underbrace{(1, 1, 1, 1)}_{u_1}, \underbrace{(0, 1, 1, 0)}_{u_2}, \underbrace{(1, 1, 0, 0)}_{u_3}\} \subseteq V$ ✓

$c_1u_1 + c_2u_2 + c_3u_3 = \vec{0}$ implies $c_1 = 0$, so $c_2 = 0$, and $c_3 = 0$. Hence S_3 is linearly indep.

Suppose $c_1u_1 + c_2u_2 + c_3u_3 = (\lambda, \mu, \eta, \lambda - \mu + \eta)$ for some $\lambda, \mu, \eta \in \mathbb{R}$

Then
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & \lambda \\ 1 & 1 & 1 & \mu \\ 1 & 1 & 0 & \eta \\ 1 & 0 & 0 & \lambda - \mu + \eta \end{array}\right] \Longrightarrow \begin{array}{l} c_1 = \lambda - \mu + \eta, \\ c_3 = \mu - \eta \\ c_2 = \mu - \lambda \end{array}$$ ✓

Hence S_3 is a basis.

(2.d) $S_4 = \{(1, 1, 1, 1), (0, 1, 1, 0), (1, 1, 0, 0), (1, 3, 4, 2)\}$

We found $\dim V = 3$, so S_4 cannot be a basis.

(3.a) Give a conceptual reason why $\{(1, 1, 3), (2, 3, 1)\}$ cannot be a basis for $\mathbb{R}^3$.

$\dim \mathbb{R}^3 = 3$, so this would violate (2.3.2).

$\mathcal{B} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis!
$\equiv \{e_1, e_2, e_3\}$

(3.b) Give a conceptual reason why $\{(1, 2, 3, 4), (2, 3, 4, 5), (1, 1, 2, 2), (4, 3, 2, 1), (12, 8, 6, 4)\}$ cannot be a basis for $\mathbb{R}^4$.

$\dim \mathbb{R}^4 = 4$, so this would violate (2.3.2)

(4) Find a basis for each vector space V and write down its dimension:

(4.a) V is the vector space of 3×3 symmetric matrices

$$V = \Big\{ a\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{v_1} + b\overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{v_2} + c\overbrace{\begin{bmatrix} 0 & & \\ & 0 & \\ & & 1 \end{bmatrix}}^{v_3} + d\overbrace{\begin{bmatrix} 0 & 1 & \\ 1 & 0 & \\ & & 0 \end{bmatrix}}^{v_4}$$
$$+ e\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{v_5} + f\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{v_6} \Big| \ a, b, c, d, e, f \in \mathbb{R} \Big\} = \operatorname{span}\{v_1, \ldots, v_6\}$$
$$= \left\{ \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \middle| \ a, b, c, d, e, f \in \mathbb{R} \right\}$$

claim: $\{v_1, v_2, v_3, v_4, v_5, v_6\} = \mathcal{B}$ is a basis!

(Supposing $c_1v_1 + \ldots + c_6v_6 = \vec{0}$ gives $\begin{bmatrix} c_1 & c_4 & c_5 \\ c_4 & c_2 & c_6 \\ c_5 & c_6 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, whence $c_1 = c_2 = \ldots = c_6 = 0$.)

$c_1 = 0$
$c_3 = 0$

(4.b) $V \subset \mathbb{R}^4$ is the solution set to

$$\text{2 eq.s}\begin{cases} x_1 + x_2 - x_3 - x_4 = 0 \\ x_1 - 2x_2 + x_3 - 2x_4 = 0 \end{cases}$$ ← we expect $\dim V = 4 - 2 = 2$

Take $x_3 = \lambda$, $x_4 = \mu$, whence $x_1 + x_2 = \lambda + \mu$, $x_1 - 2x_2 = 2\mu - \lambda$, so $x_2 = -3\mu$, and $x_1 = \lambda + 4\mu$.

$$\text{Hence } V = \left\{ \lambda \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + \mu \underbrace{\begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{v_2} \middle| \ \lambda, \mu \in \mathbb{R} \right\} = \operatorname{span}\{v_1, v_2\}.$$

Suppose $c_1v_1 + c_2v_2 = \vec{0}$. Then $c_1 = 0$ (3rd comp) and $c_2 = 0$ (4th comp), whence $\{v_1, v_2\}$ is a basis. ie. $\dim V = 2$.

Part II: Calculus

14.4

(1) Find an equation of the tangent plane (defined by $L(x,y)$, say) to the given surface at the specified point.

Recall: $L(x,y) - f(a,b) = \begin{bmatrix} f_x(a,b) \\ f_y(a,b) \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} a \\ b \end{bmatrix} \right)$, directly generalizing
$\ell(x) - f(a) = f'(a)(x - a)$, for tangent line to $y = f(x)$.
Recognize the plane analogue of the familiar form:
$y - y_1 = m(x - x_1)$.

Indeed, $\begin{bmatrix} f_x(a,b) \\ f_y(a,b) \end{bmatrix}$ is even still called the gradient, and written $\operatorname{grad}(f)\big|_{(a,b)}$ or $(\nabla f)(a,b)$ – more on this later.

(1.a) $z = \underbrace{e^{x-y}}_{f(x,y)}$; $(2, 2, 1)$ ← $1 = f(2,2)$

$$f_x(2,2) = e^{x-y}\Big|_{x=y=2} = 1, \quad f_y(2,2) = -e^{x-y}\Big|_{x=y=2} = -1$$

$$L(x,y) = 1 + 1(x-2) - 1(y-2)$$

(1.b) $z = \underbrace{xe^{xy}}_{f(x,y)}$; $(2, 0, 2)$ ← $f(2,0)$

$$f_x(2,0) = e^{xy}(1 + xy)\Big|_{\substack{x=2 \\ y=0}} = 1, \quad f_y(2,0) = x^2e^{xy}\Big|_{\substack{x=2 \\ y=0}} = 4$$

$$L(x,y) = 2 + 1(x-2) + 4(y-0)$$

(2) Explain why the function is differentiable at the given point. Then find the linearization $L(x, y)$ of the function at that given point.

Recall: $f: D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $\vec{x}_0 = (a,b) \in D_f$ if

① $\displaystyle\lim_{(x,y)\to(a,b)} \frac{|f(x,y) - L(x,y)|}{\|(x,y) - (a,b)\|}$ exists; or if

② $\displaystyle\lim_{\vec{h}\to\vec{0}} \frac{\|f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - (\nabla f)(\vec{x}_0)\cdot\vec{h}\|}{\|\vec{h}\|}$ exists (let $\vec{h} = \vec{x} - \vec{x}_0$; $\vec{x} = (x,y)$).

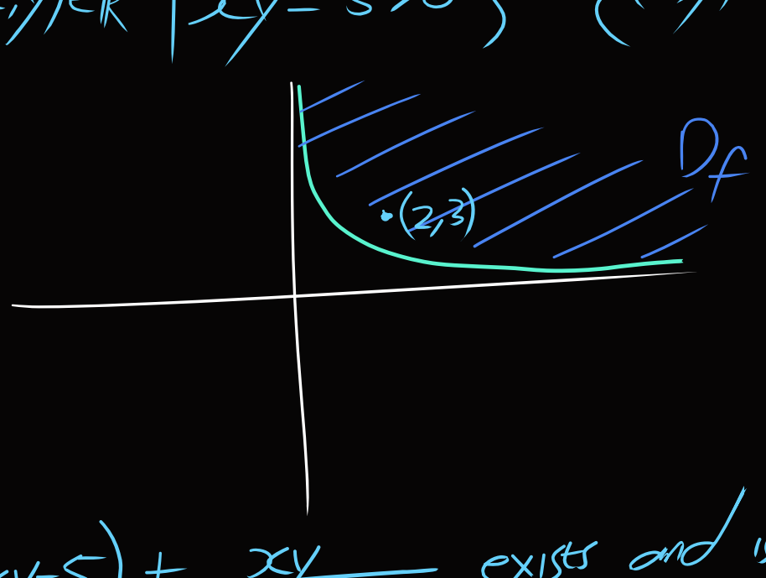
This second form generalizes very well! In fact the definition is, in general:
$f: D_f \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $\vec{x}_0 \in D_f$ if there exists a linear map $J: \mathbb{R}^m \to \mathbb{R}^n$ such that $\displaystyle\lim_{\vec{h}\to\vec{0}} \frac{\|f(\vec{x}_0 + \vec{h}) - f(\vec{x}_0) - J(\vec{h})\|}{\|\vec{h}\|}$ exists.

Thm: $f: D_f \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(a,b) \in D_f$ if
① $(a,b) \in S \subseteq D_f$ for open disk S
② f_x, f_y exist on S
③ f_x, f_y are cts at (a,b)

$f(x, y) = 1 + x \ln(xy - 5)$; $(2, 3)$

$$D_f = \{(x,y) \in \mathbb{R}^2 \mid xy - 5 > 0\} = \{(x,y) \in \mathbb{R}^2 \mid xy > 5\}.$$

$(2,3) \in D_f$



We can surround $(2,3)$ by an open disk in D_f.

$f_x = \ln(xy - 5) + \dfrac{xy}{xy - 5}$ exists and is cts on D_f
$f_y = \dfrac{x^2}{xy - 5}$ 〃 〃 on D_f

Hence f is differentiable at $(2,3)$.
$L(x,y) = \overset{1}{\cancel{f(2,3)}} + 6(x-2) + 4(y-3)$

(3) Verify the linear approximation at $(0, 0)$.

$\underbrace{e^x \cos(xy)}_{f(x,y)} \approx x + 1$

$f_x(0,0) = e^x(\cos(xy) - y\sin(xy))\Big|_{(0,0)} = 1$
$f_y(0,0) = -xe^x\sin(xy)\Big|_{(0,0)} = 0$

$f(0,0)$
$L(x,y) = 1 + 1(x - 0) + \cancel{0(y-0)} = x + 1$

And f is diff. at $(0,0)$, so well-approximated by L.

14.5

(4) Use the Chain Rule to find $\frac{dw}{dt}$ if
$w = xe^{\frac{y}{z}}$, $x = t^2$, $y = 1 - t$, $z = 1 + 2t$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$
$$= 2e^{y/z}t - \frac{x}{z}e^{y/z} - 2\frac{xy}{z^2}e^{y/z}$$
$$= \exp\left(\frac{1-t}{1+2t}\right)\left(2t - \frac{t^2}{1+2t} - \frac{2t^2(1-t)}{(1+2t)^2}\right)$$

(5) Let $p(t) = f(g(t), h(t))$, where f is differentiable. Find $p'(2)$ if
$g(2) = 4, g'(2) = -3, h(2) = 5, h'(2) = 6, f_x(4, 5) = 2, f_y(4, 5) = 8.$

$$p'(2) = f_x(g(2), h(2))\,g'(2) + f_y(g(2), h(2))\,h'(2)$$
$$= f_x(4,5)(-3) + f_y(4,5)(6) = -6 + 48 = 42$$

(6) Use the Chain Rule to find the indicated partial derivatives.

(6.b) $z = e^r \cos\theta$, $r = st$, $\theta = \sqrt{s^2 + t^2}$; $\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$
$$= e^{st}\cos\left(\sqrt{s^2+t^2}\right)t - \frac{e^{st}\sin\left(\sqrt{s^2+t^2}\right)}{2s\sqrt{s^2+t^2}}$$

$\frac{\partial z}{\partial t}$ is analogous.

(7) Consider the function
$f: \mathbb{R}^3 \to \mathbb{R}$; $(x, y, z) \mapsto xyz$.

Now, use the chain rule to determine the following derivative:

$$\frac{d}{dx}f(\underbrace{x}_{u(x)}, \overset{v(x)}{x}, \underbrace{x^2}_{w(x)}) = \frac{d}{dx}x^4 = 4x^3$$

$$\frac{d}{dx}f(u(x), v(x), w(x)) = \frac{\partial f}{\partial u}u'(x) + \frac{\partial f}{\partial v}v'(x) + \frac{\partial f}{\partial w}w'(x)$$
$$= (vw)(1) + (uw)(1) + (uv)(2x)$$
$$= x^3 + x^3 + x^2(2x)$$
$$= 4x^3$$

(8) Suppose that x and t are variables and let

$u = x + at$ and $v = x - at$

where a is a constant. Now use the chain rule to show the following: every function F of the form

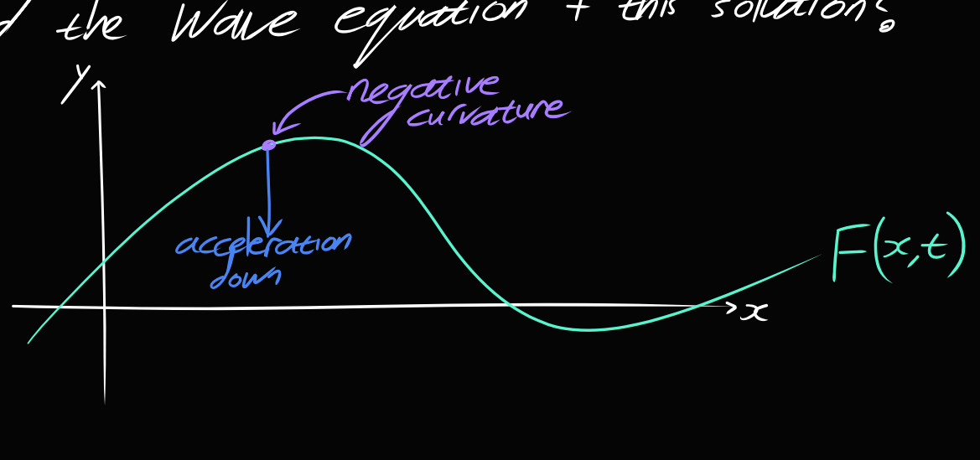
$F(x, t) = f(x + at) + g(x - at)$,

where f and g are differentiable functions of one variable, is a solution of the wave equation

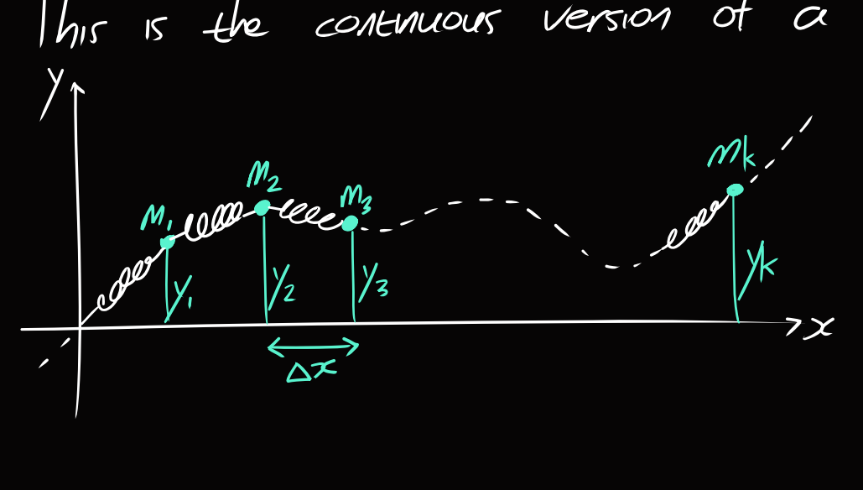
$$\frac{\partial^2 F(x,t)}{\partial t^2} = a^2\frac{\partial^2 F(x,t)}{\partial x^2}.$$

Note: How do we understand the wave equation + this solution?

$$\underbrace{F_{tt}}_{\text{vertical acceleration}} = \underset{\text{positive constant}}{a^2}\underbrace{F_{xx}}_{\text{curvature}}$$



This is the continuous version of a chain of springs:



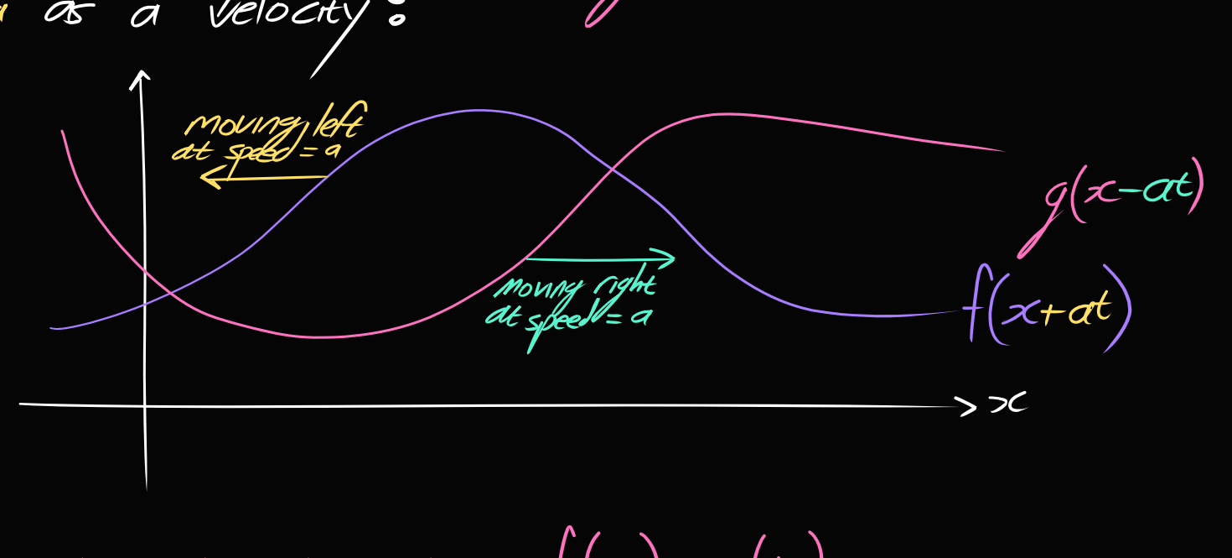
Newton's law:
$$m_k\frac{\partial^2 y_k}{\partial t^2} = \overset{\text{spring constant}}{k}\underbrace{\left[(y_{k-1} - y_k) + (y_{k+1} - y_k)\right]}_{\text{net force}}$$

To make this continuous, replace mass m_k with density, $m_k = \rho\Delta x$, spring constant k with tension, $T = k\Delta x$, and $y_k(t)$ with $y(x,t)$. Finally, take $\Delta x \to 0$:

$$\rho\frac{\partial^2 y}{\partial t^2} = T\Bigg(\underbrace{\lim_{\Delta x\to 0}\frac{y(x-\Delta x, t) + y(x+\Delta x, t) - 2y(x,t)}{\Delta x^2}}_{\text{well-known form for } y_{xx}(x,t)}\Bigg) = T\frac{\partial^2 y}{\partial x^2}$$

Define $a^2 = \frac{T}{\rho} > 0$, noting that a has units of $\sqrt{\frac{\text{Force}}{\text{Mass / Distance}}} = \text{Velocity}$.

Our solution $y = f(x + at) + g(x - at)$ matches the interpretation of a as a velocity:



$F(x, t) = f(x + at) + g(x - at), = f(u) + g(v)$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}$$
$$= f'(u)(1) + g'(v)(1)$$
$$\frac{\partial^2 F}{\partial x^2} = f''(u) + g''(v) = f''(x + at) + g''(x - at)$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial t}$$
$$= f'(u)a - g'(v)a$$
$$\frac{\partial^2 F}{\partial t^2} = f''(u)a^2 - g''(v)a(-a)$$
$$= a^2\frac{\partial^2 F}{\partial x^2}, \text{ solving wave equation}$$