Mathematics 214 Tutorial 8

Part I: Linear Algebra

§3.1

(1) Determine whether the following are linear maps:

(1.a) $T : \mathrm{Row}_3 \to \mathrm{Row}_2$ defined by $T(x,y,z) = (2x-y, x+y-z)$

① $T(\lambda(x,y,z)) = T(\lambda x, \lambda y, \lambda z) = (2\lambda x - \lambda y, \lambda x + \lambda y - \lambda z)$

$= \lambda(2x-y, x+y-z) = \lambda T(x,y,z)$ ✓

② $T((x,y,z)+(x',y',z')) = T(x+x', y+y', z+z')$

$= (2(x+x')-(y+y'), (x+x')+(y+y')-(z+z'))$

$= (2x-y, x+y-z)+(2x'-y', x'+y'-z')$

$= T(x,y,z)+T(x,y,z)$ ✓

(1.b) $T : \mathrm{Fun}(\mathbb{R},\mathbb{R}) \to \mathrm{Row}_2$ defined by $T(f) = (f(1), f(0))$

$T(\lambda f+\mu g)$ (We get lazy + check both at once)

$= ((\lambda f+\mu g)(1), (\lambda f+\mu g)(0))$

$= (\lambda f(1)+\mu g(1), \lambda f(0)+\mu g(0))$ by def of $(f+g)(x)$ & $(\lambda f)(x)$

$= \lambda(f(1), f(0)) + \mu(g(1), g(0))$

$= \lambda T(f)+\mu T(g)$ ✓

(6.a) If $T : \mathrm{Row}_2 \to \mathrm{Row}_3$ satisfies $T(1,0) = (2,1,-1)$ and $T(0,1) = (1,1,2)$, find $T(3,-2)$

$(3,-2) = 3(1,0)-2(0,1)$

$T(3,-2) = T(3(1,0)-2(0,1)) = 3T(1,0)-2T(0,1)$

$= 3(2,1,-1)-2(1,1,2)$

(6.b) If $T : \mathrm{Row}_2 \to \mathrm{Row}_4$ satisfies $T(1,1) = (1,2,3,4)$ and $T(1,-1) = (3,2,1,0)$, find $T(1,0)$ and $T(0,1)$

$(0,1) = \frac12(1,1)-\frac12(1,-1)$

$(1,0) = \frac12(1,1)+\frac12(1,-1)$

$T(1,0) = T\left(\frac12(1,1)+\frac12(1,-1)\right) = \frac12T(1,1)+\frac12T(1,-1)$

$= \frac12(1,2,3,4)+\frac12(3,2,1,0)$

(6.c) If $T : \mathrm{Poly}_2 \to \mathrm{Row}_2$ satisfies $T(1+x+x^2) = (1,2)$, $T(1+x) = (2,1)$ and $T(1) = (0,0)$, find $T(3+2x+x^2)$

$3+2x+x^2 = (1+x+x^2)+(1+x)+(1)$

$T(3+2x+x^2) = T((1+x+x^2)+(1+x)+(1))$

$= T(1+x+x^2)+T(1+x)+T(1) = (1,2)+(2,1)+(0,0)$

(7) Consider the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ of $\mathbb{R}^2$, where $\mathbf{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}1\\1\end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation satisfying

$$T(\mathbf{v}_1) = \begin{bmatrix}-1\\2\end{bmatrix} \text{ and } T(\mathbf{v}_2) = \begin{bmatrix}2\\-3\end{bmatrix}.$$

Find a formula for $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$ and use it to find $T\left(\begin{bmatrix}5\\-3\end{bmatrix}\right)$.

$\begin{bmatrix}x\\y\end{bmatrix} = y\begin{bmatrix}1\\1\end{bmatrix}+(x-y)\begin{bmatrix}1\\0\end{bmatrix}$

$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \left(y\begin{bmatrix}1\\1\end{bmatrix}+(x-y)\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}y-x\\2x-2y\end{bmatrix}+\begin{bmatrix}2y\\-3y\end{bmatrix} = \begin{bmatrix}3y-x\\2x-5y\end{bmatrix}$

(16) Let V and W be vector spaces and $T : V \to W$ a linear map. Show that if the vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ are linearly dependent in V, then the vectors $T(\mathbf{v}_1), \dots, T(\mathbf{v}_m)$ are linearly dependent in W. (Is it true that if the $\mathbf{v}_i$ are linearly independent, then the $T(\mathbf{v}_i)$ are linearly independent?) ⊛

⊛ Parenthesized question [False]

Consider $T: \mathbb{R}^2 \to \mathbb{R}^2 : \begin{bmatrix}x\\y\end{bmatrix} \mapsto \begin{bmatrix}0\\0\end{bmatrix}$. This is linear, since

$T(\lambda\underline{v}+\mu\underline{w}) = \underline{0} = \underline{0}+\underline{0} = \lambda\underline{0}+\mu\underline{0} = \lambda T(\underline{v})+\mu T(\underline{w})$.

$\exists (c_1,\dots,c_m) \neq \vec{0} \in \mathbb{R}^m : \sum_{k=1}^{m} c_k \underline{v_k} = \underline{0}_V$

Consider $T\left(\sum_{k=1}^{m} c_k \underline{v_k}\right) = T(\underline{0}_V) = T(0\cdot\underline{0}_V) = 0T(\underline{0}_V) = \underline{0}_W$

$= \sum_{k=1}^{m} c_k T(\underline{v_k}) = \underline{0}_W$

not all 0

Hence $T(\underline{v_k})$ are linearly dependent. □

§3.2

(1) Let $R_\theta : \mathrm{Col}_2 \to Col_2$ be

$$\begin{bmatrix}v_1\\v_2\end{bmatrix} \mapsto \begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Check that $R_\phi \circ R_\theta = R_{\phi+\theta}$ by computing both linear maps on an arbitrary vector $\mathsf{v} \in \mathrm{Col}_2$.

$(R_\phi\circ R_\theta)(v) = R_\phi\left(\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}\right) = \begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$

$= \begin{bmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{bmatrix}\begin{bmatrix}\cos\theta\, v_1+\sin\theta\, v_2\\ \cos\theta\, v_2-\sin\theta\, v_1\end{bmatrix}$

sin double ∠

$= \begin{bmatrix}\cos\phi\cos\theta\, v_1+\cos\phi\sin\theta\, v_2+\sin\phi\cos\theta\, v_2-\sin\phi\sin\theta\, v_1\\ -\sin\phi\cos\theta\, v_1-\sin\phi\sin\theta\, v_2+\cos\phi\cos\theta\, v_2-\cos\phi\sin\theta\, v_1\end{bmatrix}$ cos double ∠

$= \begin{bmatrix}\cos(\phi+\theta) & \sin(\phi+\theta)\\ -\sin(\phi+\theta) & \cos(\phi+\theta)\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = R_{\phi+\theta}(v)$ ✓ $R_\phi\circ R_\theta = R_{\phi+\theta}$

(2) Let $M : \mathrm{Poly}_3 \to \mathrm{Poly}_4$ be $M(p)(x) = xp(x)$. Let $S : \mathrm{Poly}_4 \to \mathrm{Poly}_4$ be the map $S(p)(x) = p(x-1)$. Similarly let $T : \mathrm{Poly}_3 \to \mathrm{Poly}_3$ be the map $T(p)(x) = p(x-1)$. Compute $S\circ M$ and $M\circ T$. Are they equal?

$(S\circ M)(p(x)) = S(M(p(x))) = S(xp(x)) = (x-1)p(x-1)$

$(M\circ T)(q(x)) = M(T(q(x))) = M(q(x-1)) = x\,q(x-1)$

$S\circ M \neq M\circ T$, since $(S\circ M)(x\mapsto 1) = (x\mapsto x-1)$

but $(M\circ T)(x\mapsto 1) = (x\mapsto x)$

not equal (at $x=0$, say)

§3.3

(1) Are the following vector spaces isomorphic?

$$V = \left\{\mathsf{v} \in \mathrm{Col}_4 : \begin{bmatrix}1 & 2 & 0 & -1\\ -1 & 1 & 1 & 0\end{bmatrix}\mathsf{v} = 0\right\}$$

$$W = \left\{p \in \mathrm{Poly}_2 : \int_0^2 p(x)dx = 0\right\}.$$

If they are, construct an explicit isomorphism between them. If not, prove that they are not isomorphic.

Exploration:

① Row-reduced augmented matrix $= \left[\begin{array}{cccc|c}1 & 2 & 0 & -1 & 0\\ 0 & 3 & 1 & -1 & 0\end{array}\right]$

$V = \left\{\lambda\begin{bmatrix}-2\\1\\3\\0\end{bmatrix}+\mu\begin{bmatrix}5\\-1\\0\\3\end{bmatrix} \,\middle|\, \lambda,\mu\in\mathbb{R}\right\}$ ← $\dim V = 2$, clearly

basis

② $\int_0^2 (p_0+p_1x+p_2x^2)\,dx = p_0(2)+\frac12p_1(2)^2+\frac13p_2(2)^3 = 0$

$W = \left\{\lambda(-1+x)+\mu(-4+3x^2) \,\middle|\, \lambda,\mu\in\mathbb{R}\right\}$ ← $\dim W = 2$ also

Define $T: V\to W$ by $T\begin{bmatrix}-2\\1\\3\\0\end{bmatrix} := -1+x$

and $T\begin{bmatrix}5\\-1\\0\\3\end{bmatrix} := -4+3x^2$, (and make it linear)

ie. $T\left(\lambda\begin{bmatrix}-2\\1\\3\\0\end{bmatrix}+\mu\begin{bmatrix}5\\-1\\0\\3\end{bmatrix}\right) = \lambda(-1+x)+\mu(-4+3x^2)$

T is invertible with linear inverse, so $V\cong W$

(2) Are the following vector spaces isomorphic?

$$V = \{\mathbf{v}\in\mathbb{R}^3 : \mathbf{v}\times(1,2,3) = \mathbf{0}\}$$

$$W = \{M \in \mathrm{Mat}_{2,2} : M^T = -M\}.$$

If they are, construct an explicit isomorphism between them. If not, prove that they are not isomorphic.

$V = \{k(1,2,3) \mid k\in\mathbb{R}\}$, $W = \left\{\begin{bmatrix}0 & k\\ -k & 0\end{bmatrix} \,\middle|\, k\in\mathbb{R}\right\}$

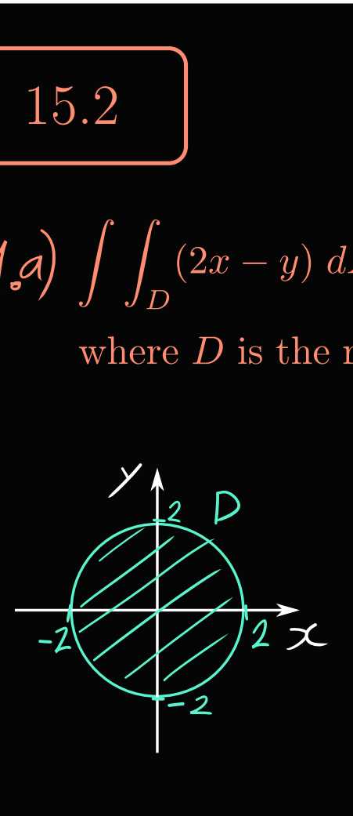
linear $T: V\to W$, defined $(1,2,3)\longmapsto\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$ is an isomorphism,

so $V\cong W$

Part II: Calculus

15.2

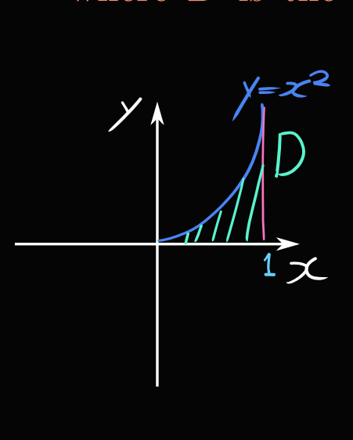
(1.a) $\iint_D (2x-y)\,dA$, where D is the region bounded by the circle with center the origin and radius 2. $x^2+y^2=4$



$\iint_D (2x-y)\,dA = \begin{cases} ① \int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} (2x-y)\,dx\,dy & (\mathrm{II})\\ ② \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-y)\,dy\,dx & \leftarrow \text{we use this one } (\mathrm{I})\\ ③ \int_0^{2\pi}\int_0^2 (2r\cos\theta - r\sin\theta)\,r\,dr\,d\theta \end{cases}$

$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x-y)\,dy\,dx = \int_{-2}^{2}(\underbrace{4x\sqrt{4-x^2}}_{\text{odd func}})\,dx = 0$

(1.b) $\iint_D x\cos y\,dA$, where D is the region bounded by $y = 0$, $y = x^2$ and $x = 1$.

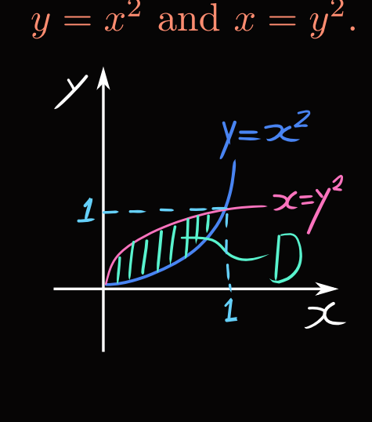


$\int_0^1\int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 x\sin(x^2)\,dx$ $u=x^2$

$= \int_0^1 \frac12\sin(u)\,du = \frac12\sin(1)$

(2) Find the volume of:

(2.a) the solid under the plane $3x+2y-z=0$ and above the region enclosed by the parabolas $y=x^2$ and $x=y^2$.



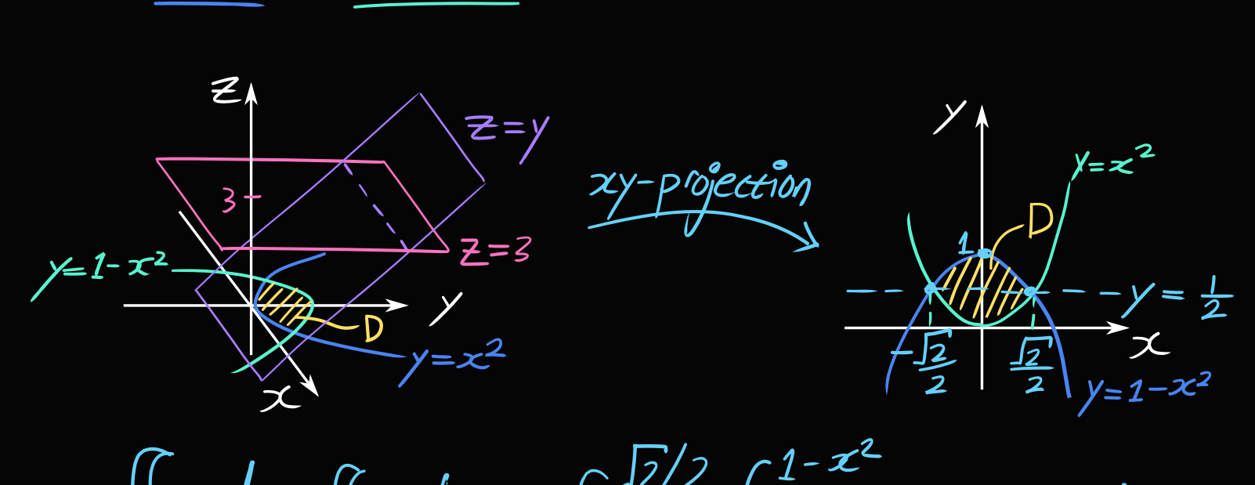
$\iint_D (-3x-2y)\,dA = \int_0^1\int_{x^2}^{\sqrt{x}} (-3x-2y)\,dy\,dx$

$= \int_0^1 \left(-3x(\sqrt{x}-x^2)-(x-x^4)\right)dx$

$= \int_0^1 \left(-3x^{3/2}+3x^3-x+x^4\right)dx$

$= -\frac65+\frac34-\frac12+\frac15$

(2.b) the solid under the plane $z=3$, above the plane $z=y$ and between the parabolic cylinders $y=x^2$ and $y=1-x^2$. Find this volume by subtracting two volumes.

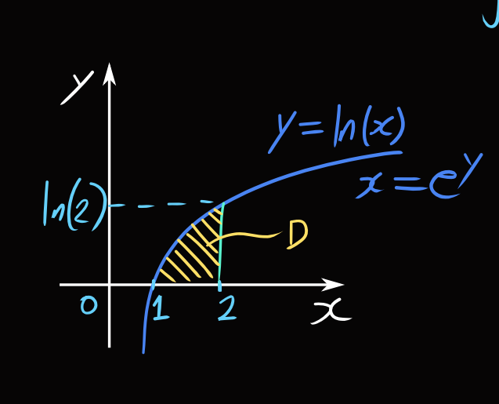


$\iint_D 3\,dA - \iint_D y\,dA = \int_{-\sqrt2/2}^{\sqrt2/2}\int_{x^2}^{1-x^2} (3-y)\,dy\,dx$

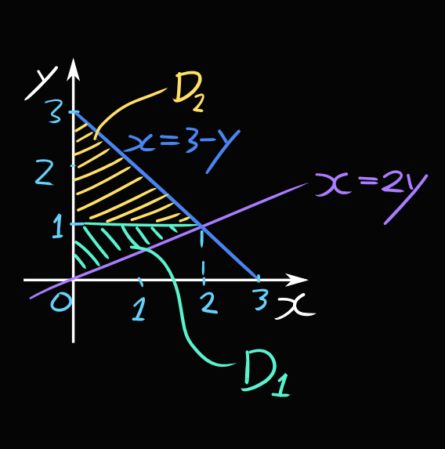
$= \int_{-\sqrt2/2}^{\sqrt2/2} \left(3(1-2x^2)-\frac12\left((1-x^2)^2-x^4\right)\right)dx = \dots$

(3) Sketch the region of integration and change the order of integration. Also evaluate the integrals in (c) and (d).

(3.a) $\int_1^2\int_0^{\ln x} f(x,y)\,dydx = \int_0^{\ln(2)}\int_{e^y}^{2} f(x,y)\,dx\,dy$

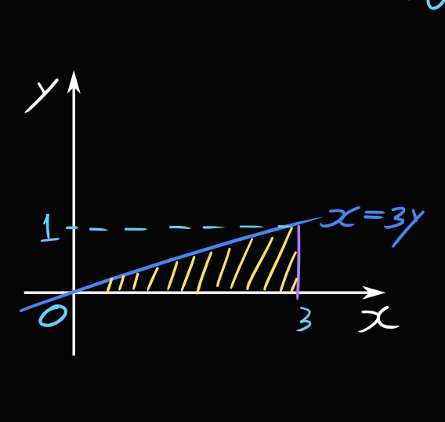


(3.b) $\iint_D f(x,y)\,dA = \underbrace{\int_0^1\int_0^{2y} f(x,y)\,dxdy}_{D_1} + \underbrace{\int_1^3\int_0^{3-y} f(x,y)\,dxdy}_{D_2}$



$= \int_0^2\int_{\frac12 x}^{3-x} f(x,y)\,dy\,dx$

(3.c) $\int_0^1\int_{3y}^3 e^{x^2}\,dxdy = \int_0^3\int_0^{\frac13 x} e^{x^2}\,dy\,dx$

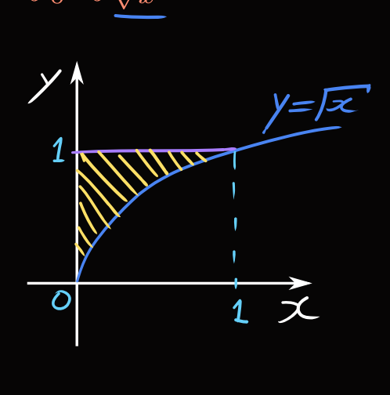


$= \int_0^3 \frac13 x\, e^{x^2}\,dx$ $u=x^2$

NB! Switching order made integral possible

$= \int_0^9 \frac16 e^u\,du = \frac16(e^9-1)$

(3.d) $\int_0^1\int_{\sqrt{x}}^1 \sqrt{y^3+1}\,dydx = \int_0^1\int_0^{y^2} \sqrt{y^3+1}\,dx\,dy$



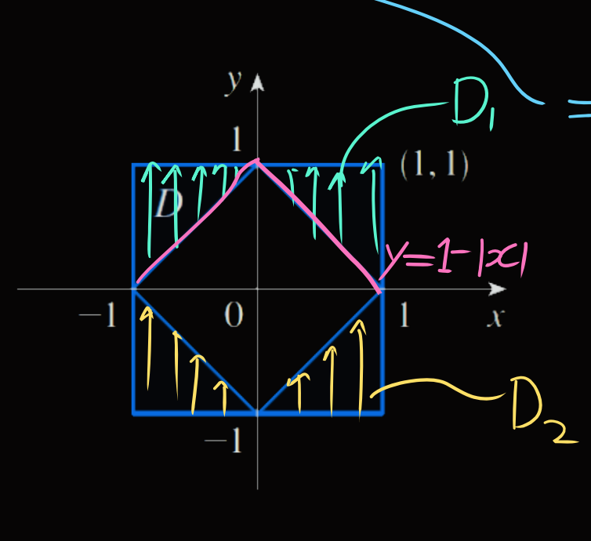
$= \int_0^1 y^2\sqrt{y^3+1}\,dy$ $u=y^3+1$

$= \int_0^2 \frac13\sqrt{u}\,du$

$= \frac29 2^{3/2}$

(4) Express D as a union of regions of type I and type II and evaluate the integral.

(4.a) $\iint_D x^2dA$ with



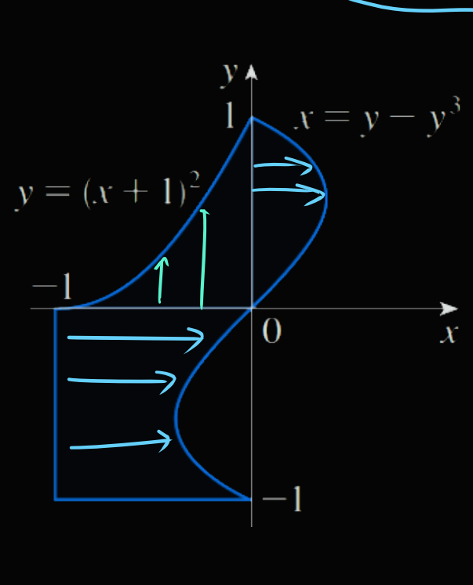
$= \iint_{D_1} x^2dA + \iint_{D_2} x^2dA$

$= \int_{-1}^1\left(\int_{1-|x|}^1 x^2\,dy + \int_{-1}^{|x|-1} x^2\,dy\right)dx$

$= \int_{-1}^1 2|x|x^2\,dx = \int_{-1}^0 (-2x^3)\,dx + \int_0^1 2x^3\,dx$

$= 2\int_0^1 2x^3dx = 1$

(4.b) $\iint_D y\,dA$ with



$\int_0^1 \boxed{\int_0^{y-y^3} y\,dx}\,dy + \int_{-1}^0\int_0^{(x^2+1)} y^2\,dy\,dx$ ← y^2-y^4

$+ \int_{-1}^0 \boxed{\int_{-1}^{y-y^3} y\,dx}\,dy$ ← y^2-y^4+y

$= \left(\frac13-\frac15\right)-\left(-\frac13+\frac15+\frac12\right)$

$+ \int_{-1}^0 \frac13(x^2+1)^3\,dx = \dots$